Birth Death Processes - Additional Notes

Tom Bury

May 14, 2017

1 Extinction

Deterministic modelling fails to capture the possibility of extinction, which is of significant importance to numerous areas of science. Under the more realistic, stochastic framework, we may investigate probabilities of extinction and in turn take measures to avoid (e.g. an endangered species) or promote (e.g. infectious disease) such an event.

Assignment 3 investigates the probability of extinction for a population governed by Malthus' Law. Below we show that extinction is inevitable for any non-adapting population i.e. birth and death rates are only state (not time) dependent.

Let $b(n)$ and $d(n)$ represent the birth and death rate and $\xi(n) = \frac{d(n)}{b(n)}$ be the relative death to birth rate. We assume the following:

- $b(0) = 0$: The ground state is absorbing (no immigration)
- $d(n) > 0$: It is always possible to die
- $\xi(n) \to \infty$ as $n \to \infty$: A limiting factor stops the population increasing without bound

Let $p_e(n)$ be the probability of extinction given n individuals. Then $p_e(0) = 1$ and

$$p_e(n) = P(\text{birth next})p_e(n+1) + P(\text{death next})p_e(n-1) \tag{1}$$

$$= \frac{b(n)}{b(n)+d(n)}p_e(n+1) + \frac{d(n)}{b(n)+d(n)}p_e(n-1) \tag{2}$$

from which we can show

$$p_e(n+1) - p_e(n) = \xi(n)\left[p_e(n) - p_e(n-1)\right] \tag{3}$$

and using recurrence, we get

$$\underbrace{p_e(n+1)-p_e(n)}_{\in[-1,1]} = \underbrace{\prod_{k=1}^{n} \xi(k)}_{\substack{\text{increases}\\ \text{without bound}}} \underbrace{[p_e(1)-p_e(0)]}_{\text{so this }=0}. \tag{4}$$

Since $p_e(1) = p_e(0) = 1$, we have $p_e(n) = 1\ \forall n$, i.e eventual extinction is certain!

2 Simulation of Birth-Death Processes

Consider a temporally homogeneous birth-death process with microscopic transition rates

$$\omega(n+1|n) = b(n), \qquad \omega(n-1|n) = d(n). \tag{5}$$

For simple systems involving linear birth/death rates, we may invoke the macroscopic rate equations to pin down how the moments of the distribution evolve. However, for more complicated systems typically involving non-linear terms, the macroscopic equations may not form a closed system, and therefore we turn to simulation via a rigorously defined algorithm:

2.1 Gillespie's Algorithm

Since its derivation in the classic text [1], Gillespie's algorithm has been widely applied, with particular focus on chemical and biological systems. Upon each iteration, it relies upon the generation of two random variables. One determines the time until the next event which we refer to as *departure time* denoted τ. The other decides which event is to occur.

2.2 Distribution of Departure time

To randomly generate the departure time, we must know its probability distribution. We introduce the probability $q(n,t;\tau)$ that the system in state $n(t)$ will jump out of this state at an instant between t and $t+\tau$. By temporal homogeneity, this is in fact independent of t, though may easily be generalised to non-homogeneous processes [2]. Over an infinitesimal interval $d\tau$ we have

$$q(n;d\tau) = \sum_{n'} \omega(n'|n)d\tau = r(n)d\tau \tag{6}$$

where $r(n) = b(n) + d(n)$ is the total microscopic rate of leaving the state n.

The probability that the system does not leave the state n in this infinitesimal interval is then

$$q^*(n;d\tau) = 1 - q(n;d\tau).$$

Since the process is memoryless (i.e Markovian), we have

$$q^*(n;\tau+d\tau) = q^*(n;\tau)q^*(n,d\tau) \tag{7}$$

$$= q^*(n;\tau)\left(1 - r(n)d\tau\right) \tag{8}$$

which gives the differential equation

$$\frac{dq^*}{d\tau} = -r(n)q^*(n,\tau). \tag{9}$$

This is easily solved to give

$$q^*(n;\tau) = \exp(-r(n)\tau) \tag{10}$$

noting that the probability of not having left in time zero, $q^*(n;0) = 1$.

The probability that the system jumps out of state $n(t)$ exactly after an elapsed time of τ is then

$$\underbrace{q^*(n,\tau)}_{\text{no jump in } [0,\tau)} \times \underbrace{r(n)d\tau}_{\text{jump in } [\tau,\tau+d\tau]} \tag{11}$$

giving a p.d.f of

$$f_T(\tau) = r(n)e^{-r(n)\tau} \tag{12}$$

telling us that the departure time is *exponentially distributed*, i.e. $T \sim \text{Exp}(r(n))$

In a simulation, this may be generated from a unit uniform random variable $u_1 \in U(0,1)$ using the inversion relation[1]

$$\tau = \frac{1}{r(n)} \ln\left(\frac{1}{u_1}\right). \tag{13}$$

2.3 Event Selection

The probability of a particular event μ occurring given that some event has already happened is proportional to its transition rate which we label ω_μ. Normalising using the sum over all rates $r(n)$ we have

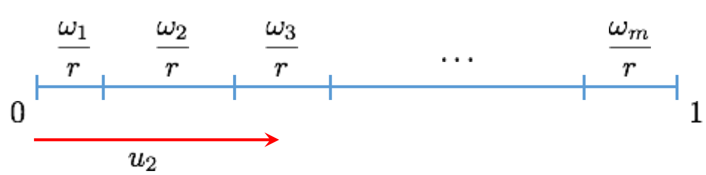
$$P(\mu \,|\text{an event happens}) = \frac{\omega_\mu(n)}{r(n)} \tag{14}$$

So, in our simple case where only two events may occur (birth and death) we have

$$P\left(\text{birth} \,|\text{an event happens}\right) = \frac{b(n)}{r(n)}, \qquad P\left(\text{death} \,|\text{an event happens}\right) = \frac{d(n)}{r(n)}. \tag{15}$$

An outcome is generated by splitting the unit interval into sections proportional to each rate and selecting one based on where a unit random variable lands. Equivalently, we select event μ that is the first integer for which

$$\frac{1}{r(n)} \sum_{j=1}^{\mu} \omega_j(n) > u_2. \tag{16}$$



In the case of the BD process then, birth is chosen if

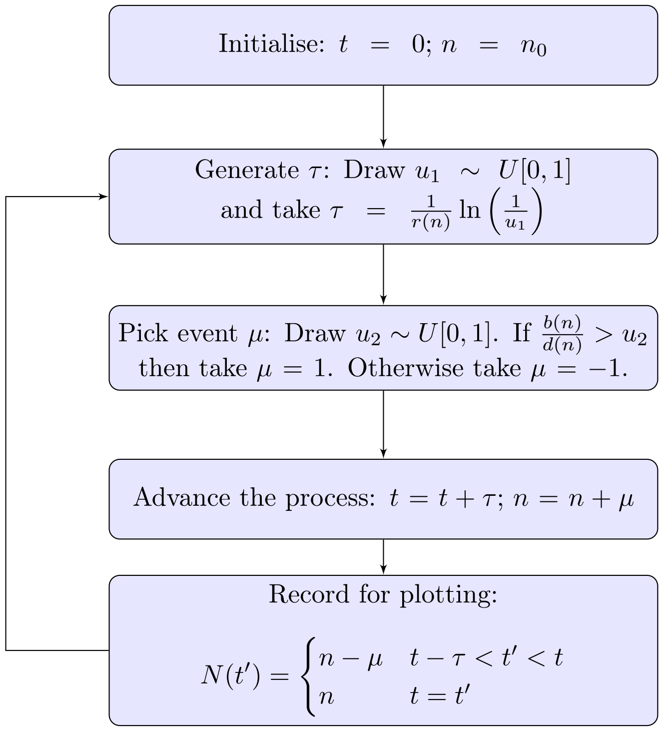
$$\frac{b(n)}{r(n)} > u_2, \tag{17}$$

otherwise death is selected. The time and state of the system are then updated and the algorithm continues until some termination criterion.

[1] Recall from probability theory that we may set the c.d.f (in this case $F_T(\tau) = 1 - e^{-r\tau}$) equal to a unit r.v. to derive the inversion relation that generates random variables of that particular distribution.

2.4 Pseudocode for Gillespie's Algorithm

The algorithm may be summarised with the following pseudocode. It produces the stochastic time series $N(t)$ given the initial condition $N(0) = n_0$.



References

[1] Daniel T Gillespie. Exact stochastic simulation of coupled chemical reactions. *The journal of physical chemistry*, 81(25):2340–2361, 1977.

[2] Daniel T Gillespie. *Markov processes: an introduction for physical scientists*. Elsevier, 1991.