Spectral early warning signals improve tipping point detection and description

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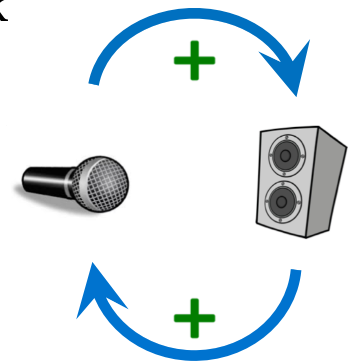
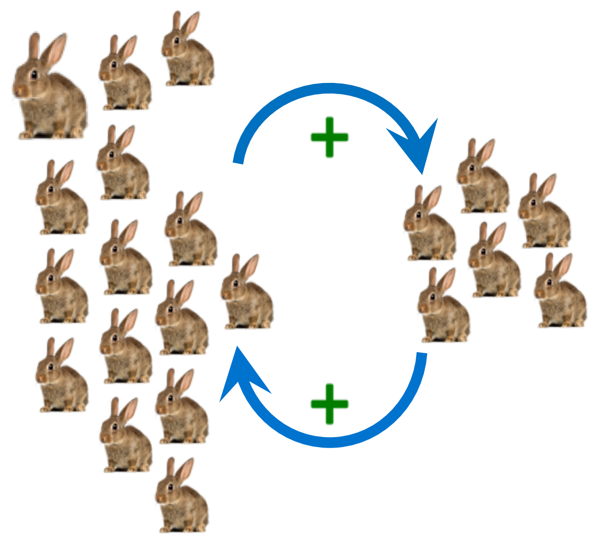
Tipping Point

"any situation where accelerating change caused by a positive feedback drives the system to a new state."

Van Nes et al. **What do you mean, 'tipping point?** (2016)

Positive feedback





Alternate stable states

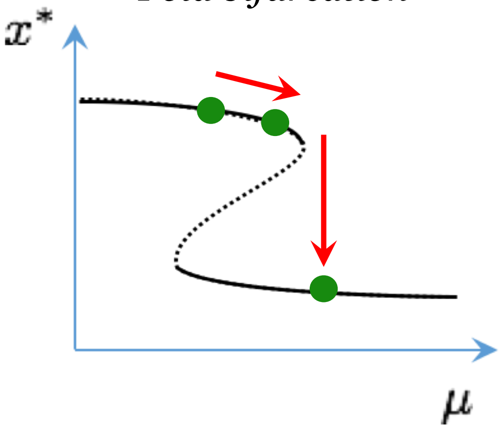
Tipping Point

Motivation




Mathematical principles

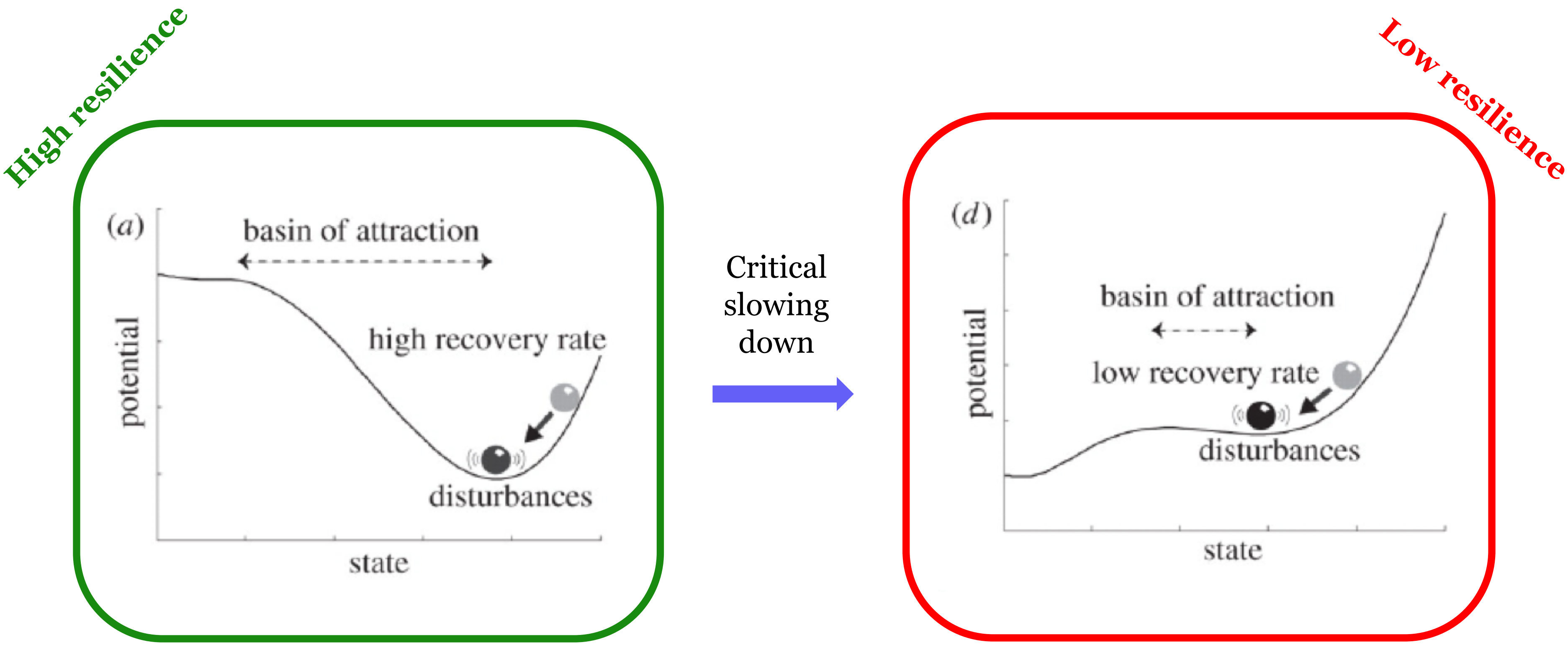
Fold bifurcation



- Positive feedback
- Alternate stable states
- Hysteresis

Early warning signals

- Statistical metrics to warn of an approaching transition

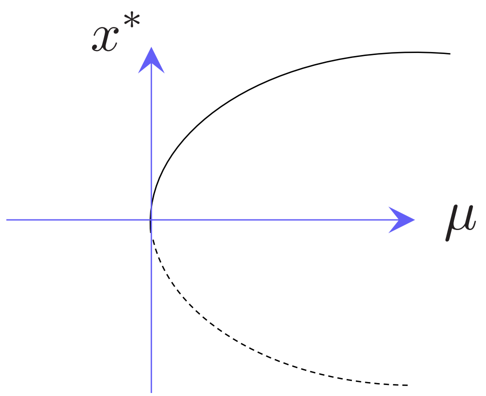


Scheffer et al. **Early-warning signals for critical transitions** (Nature, 2009)

Early warning signals

- A simple system for illustration

Normal form of the Fold bifurcation



$$\dot{x} = \mu - x^2 + \sigma\epsilon(t)$$

$$x^* = \sqrt{\mu}$$ *Stable state*

$$y = x - x^*$$ *Residual dynamics about stable state…*

satisfy

$$\dot{y} = -2\sqrt{\mu}y + \sigma\epsilon(t) + O(y^2)$$

Upon linearization

$$\dot{y} = \lambda y + \sigma\epsilon(t)$$ an **Ornstein-Uhlenbeck Process**

Early warning signals

$$\dot{y} = \lambda y + \sigma\epsilon(t)$$

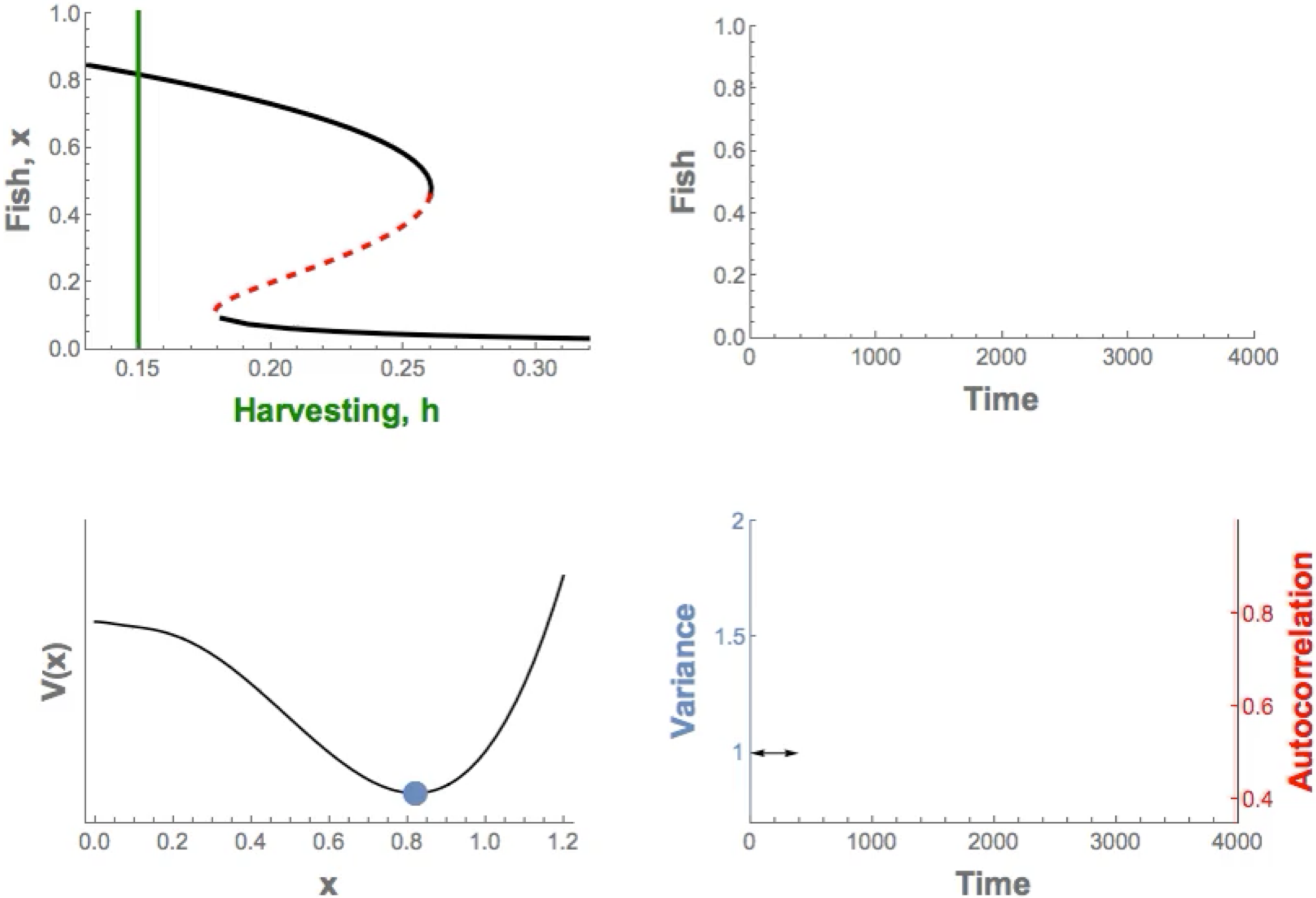
- The Ornstein Uhlenbeck Process has well-known statistical properties (e.g Gardiner 1985)

$$\mathrm{Var}(y) = \frac{\sigma^2}{2|\lambda|} \qquad \to \infty \quad \text{as} \quad |\lambda| \to 0$$

$$\phi_y(\tau) = e^{-|\lambda|\tau} \qquad \to 1 \quad \text{as} \quad |\lambda| \to 0$$

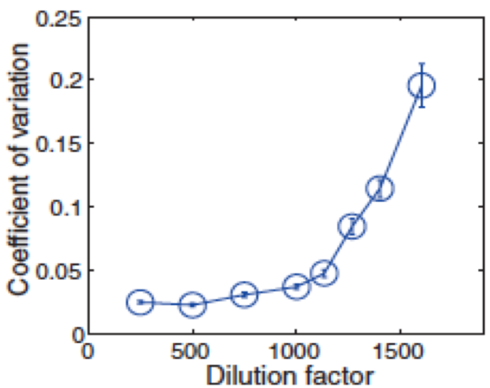
Gardiner, Handbook of stochastic methods (1985)

Early warning signals – in action



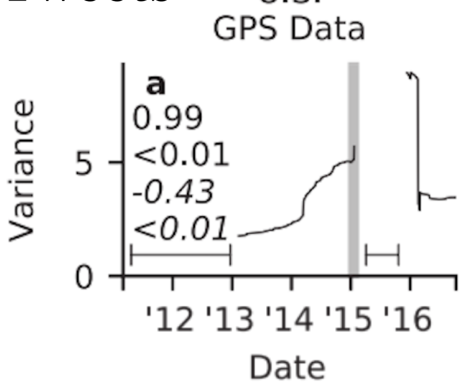
Early warning signals - observations

Microbial Experiments



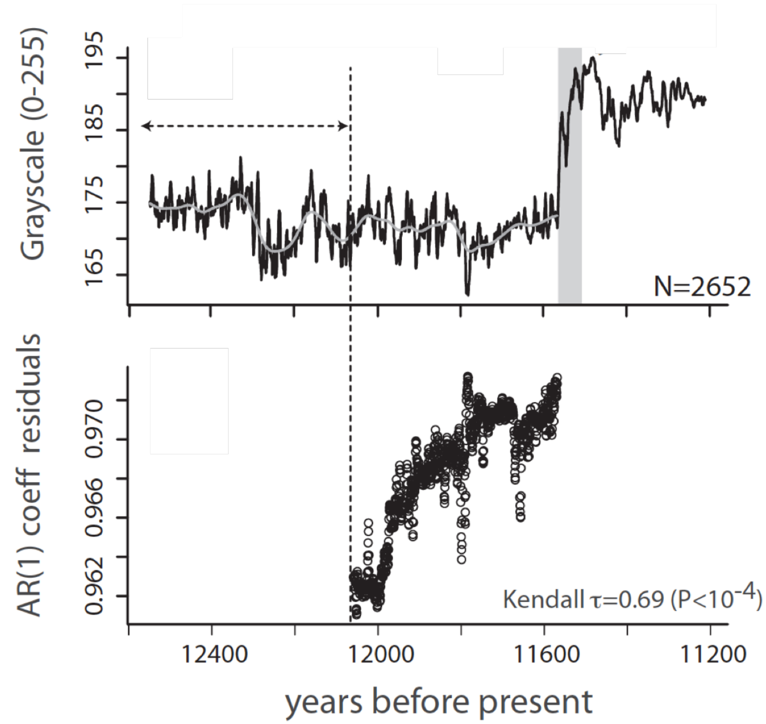
Dai et al. Science (2012)

Tweets



Pananos et al. PNAS (2017)

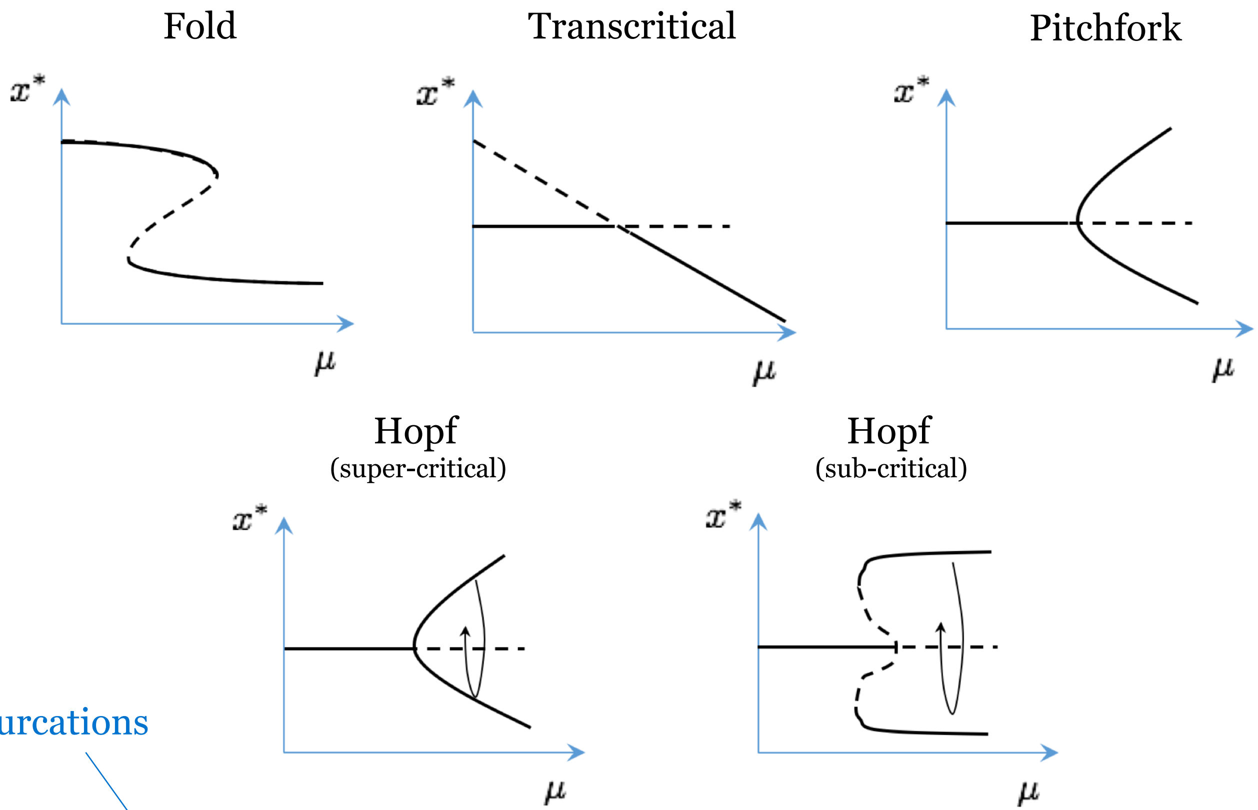
Paleoclimate records



Dakos et al. PNAS (2008)

However, bifurcations come in many flavours

E.g



Local bifurcations

and are all accompanied by critical slowing down – the mechanism that generates standard EWS (variance, autocorrelation).

Kefi et al. **Early warning signals also precede non-catastrophic transitions.** Wiley (2013)

Analytical forms preceding each normal form bifurcation

Continuous-time

Discrete-time

Bifurcation	Dominant eigenvalue(s)	Variance	Lag-τ AC, $\rho(\tau)$	Power Spectrum, $S(\omega)$
Fold, Transcritical, Pitchfork	$\lambda \in \mathbb{R}, \quad \lambda \to 0^-$	$-\frac{\sigma^2}{2\lambda}$	$e^{\lambda\|\tau\|}$	$\frac{\sigma^2}{2\pi}\left(\frac{1}{\omega^2+\lambda^2}\right)$
Hopf (super/sub-critical)	$\lambda_{1,2} = \mu \pm i\omega_0, \quad \mu \to 0^-$	$-\frac{\sigma_1^2}{2\mu}$	$e^{\mu\|\tau\|}\cos\omega_0\tau$	$\frac{\sigma_1^2}{4\pi}\left(\frac{1}{(\omega-\omega_0)^2+\mu^2} + \frac{1}{(\omega+\omega_0)^2+\mu^2}\right)$
Fold, Transcritical, Pitchfork	$\lambda \in \mathbb{R}, \quad \lambda \to 1^-$	$\frac{\sigma^2}{1-\lambda^2}$	$\lambda^{\|\tau\|}$	$\frac{\sigma^2}{2\pi}\left(\frac{1}{1+\lambda^2-2\lambda\cos(\omega)}\right)$
Flip	$\lambda \in \mathbb{R}, \quad \lambda \to -1^+$	$\frac{\sigma^2}{1-\lambda^2}$	$\lambda^{\|\tau\|}$	$\frac{\sigma^2}{2\pi}\left(\frac{1}{1+\lambda^2-2\lambda\cos(\omega)}\right)$
Neimark-Sacker	$\lambda_{1,2} = re^{\pm i\theta}, \quad r \to 1^-$	$\frac{\sigma_1^2}{1-r^2}$	$r^{\|\tau\|}\cos(\theta\tau)$	$\frac{\sigma_1^2}{4\pi}\left(\frac{1}{1+r^2-2r\cos(\omega-\theta)} + \frac{1}{1+r^2-2r\cos(\omega+\theta)}\right)$

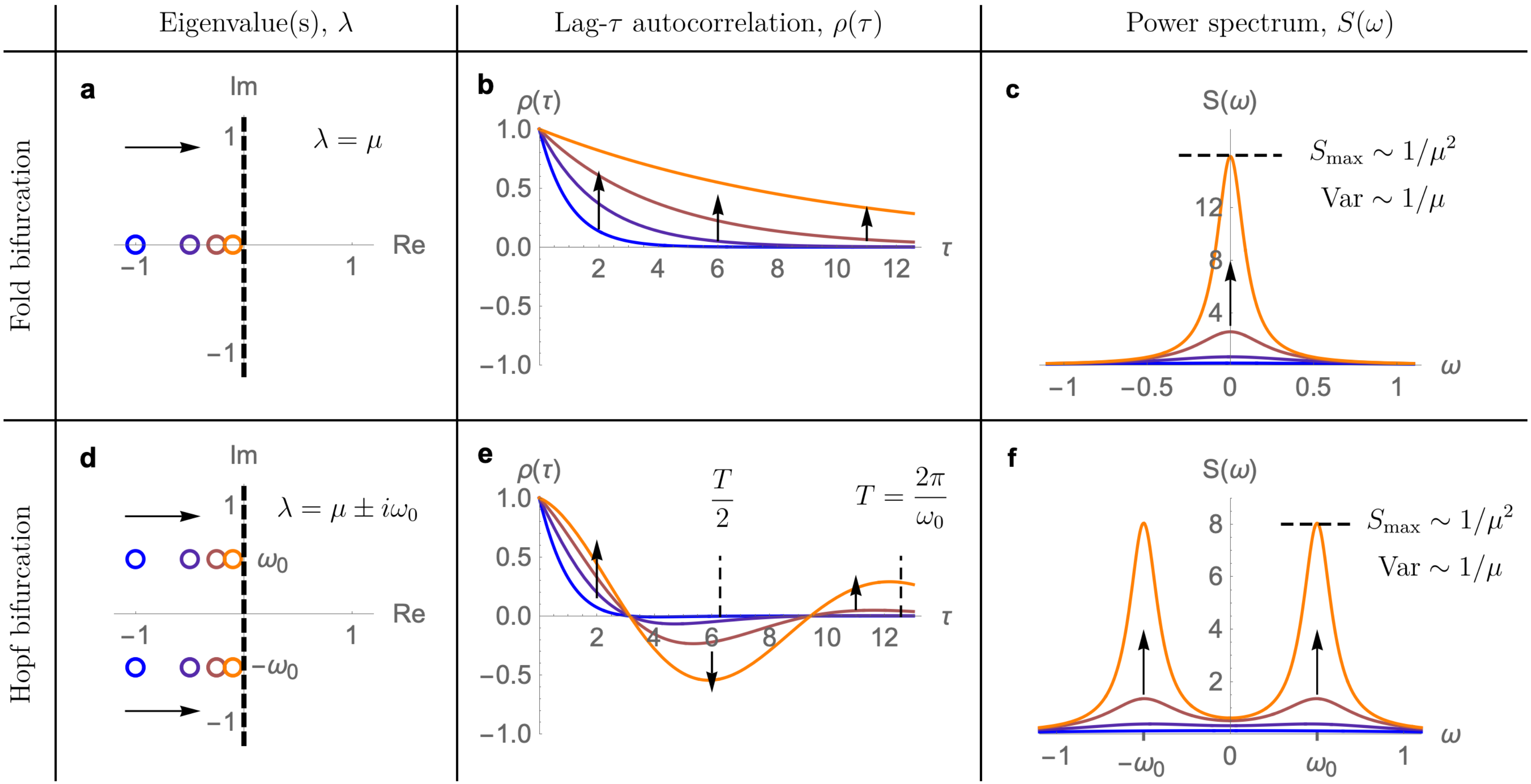
Wiesenfeld, Noisy precursors of nonlinear instabilities (1985)

Kuehn, A mathematical framework for critical transitions: normal forms, variance and applications (2013)

O'Regan et al, How Stochasticity Influences Leading Indicators of Critical Transitions (2018)

Interesting differences among bifurcations

E.g Hopf vs. Fold bifurcation



Harnessing information in the power spectrum

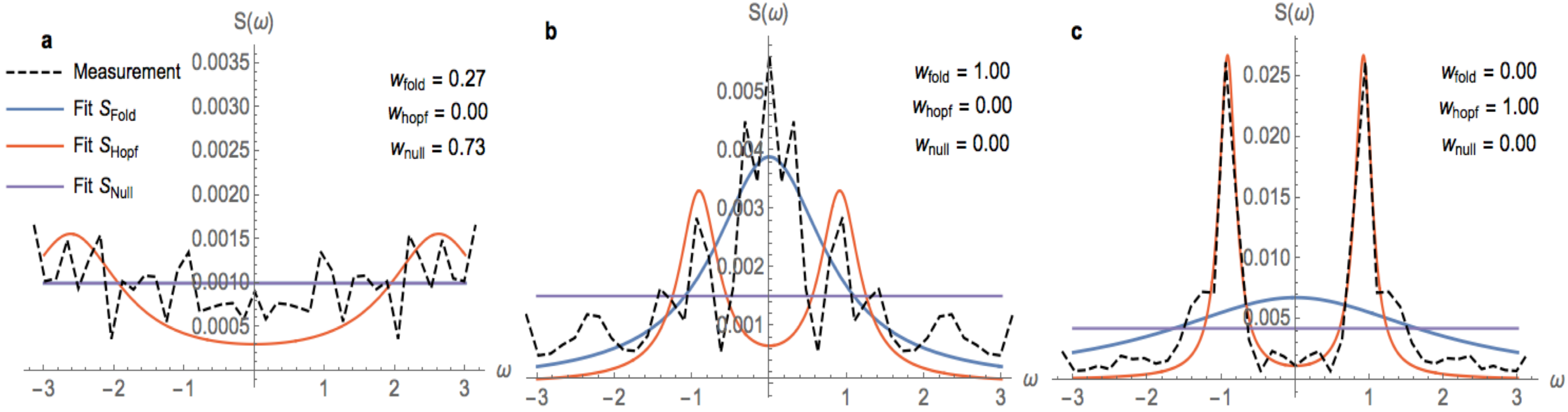
1. The shape: Fit analytical forms

$$S_{\mathrm{Null}}(\omega) = \frac{\sigma^2}{2\pi} \qquad S_{\mathrm{Fold}}(\omega) = \frac{\sigma^2}{2\pi}\frac{1}{\omega^2+\lambda^2} \qquad S_{\mathrm{Hopf}}(\omega) = \frac{\sigma^2}{2\pi}\frac{1}{2}\left\{\frac{1}{(\omega+\omega_0)^2+\mu^2}+\frac{1}{(\omega-\omega_0)^2+\mu^2}\right\}$$

to the measured power spectrum, and compute **AIC weights**

$$w_{\mathrm{null}} \qquad w_{\mathrm{fold}} \qquad w_{\mathrm{hopf}}$$

which measure goodness of fit.



Information in the power spectrum

2: The maximum power

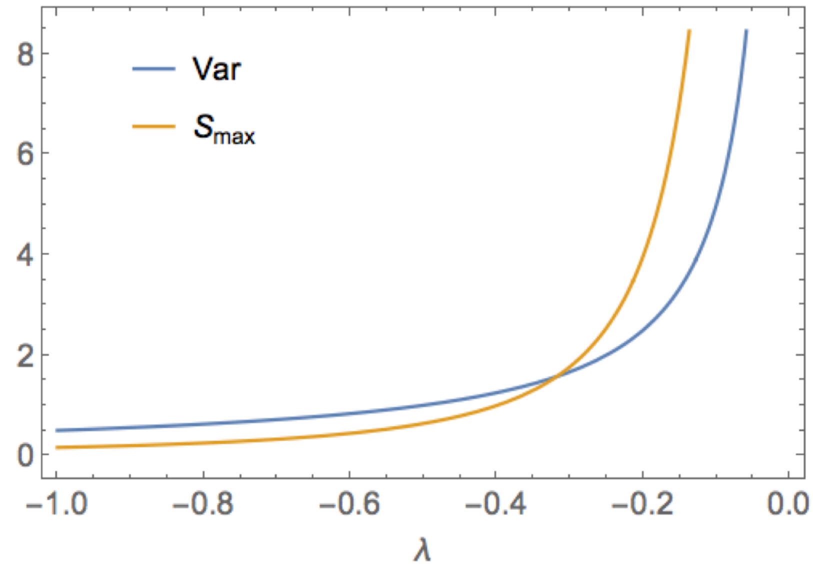
When d is halved, Var increases two-fold, whereas Smax increases 4-fold.

Maximum power (Smax)

$$S_{\max} \sim \frac{1}{d^2}$$

Variance

$$\mathrm{Var} \sim \frac{1}{d}$$



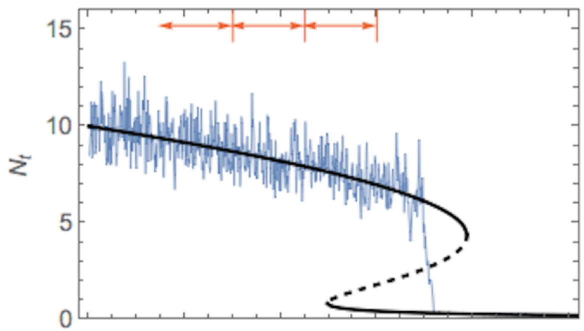
Together: Smax detects change in bifurcation proximity, AIC weights detect bifurcation class.

Ricker-type population model

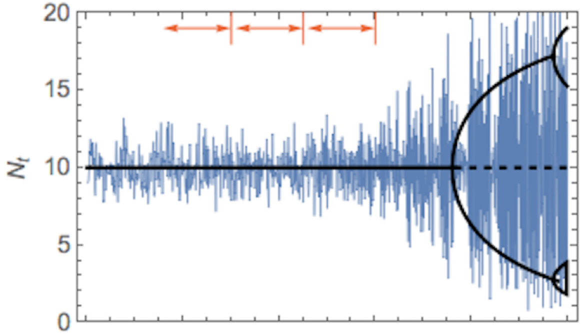
$$N_{t+1} = N_t e^{(r_t(1-N_t/K)+\sigma\epsilon_t)} - F\frac{N_t^2}{N_t^2 + h^2}$$

Ricker. **Stock and recruitment** Journal of the Fisheries Board of Canada (1954)

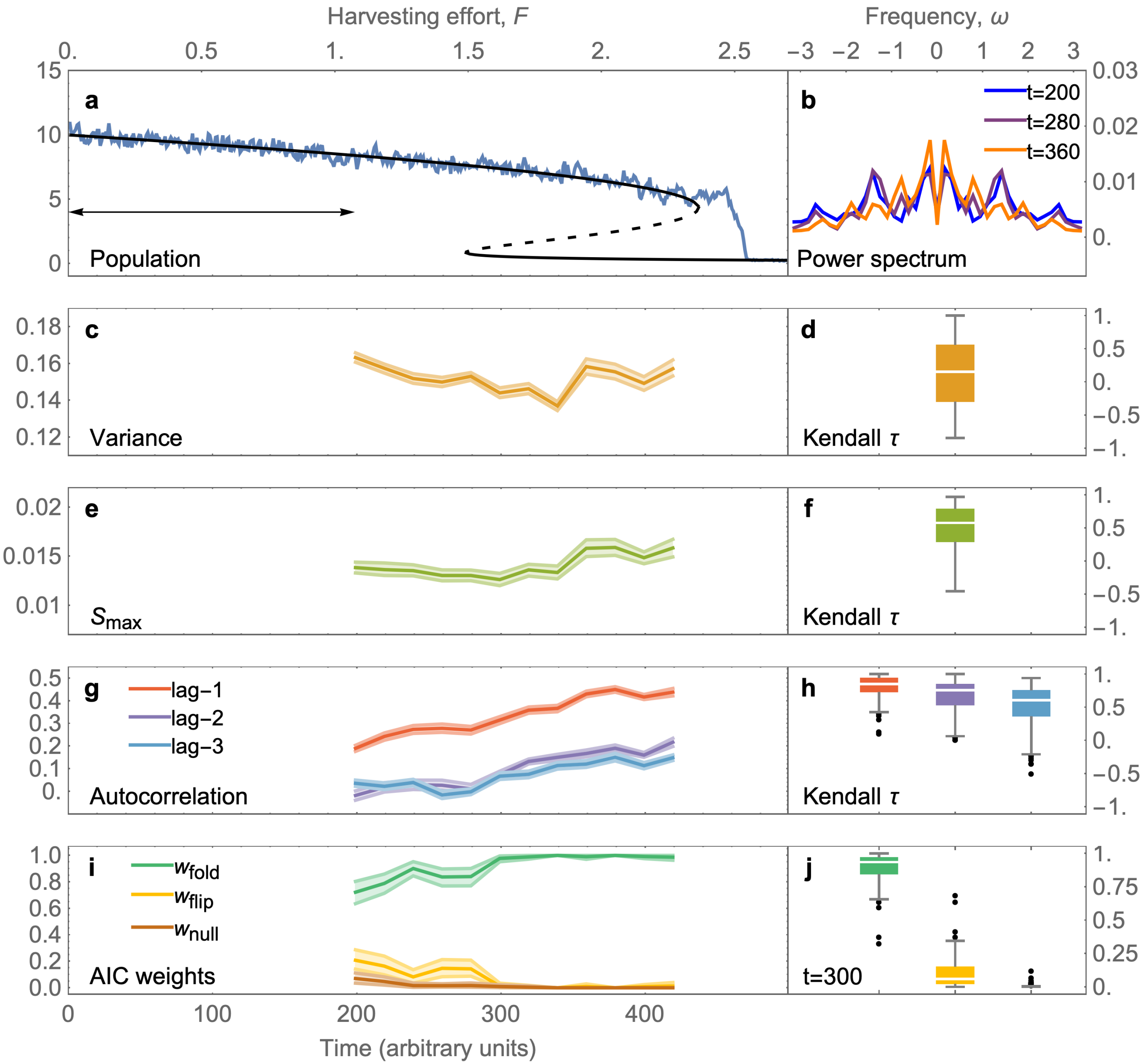
- Two possible bifurcations from the stable equilibrium.
- Both preceded by CSD



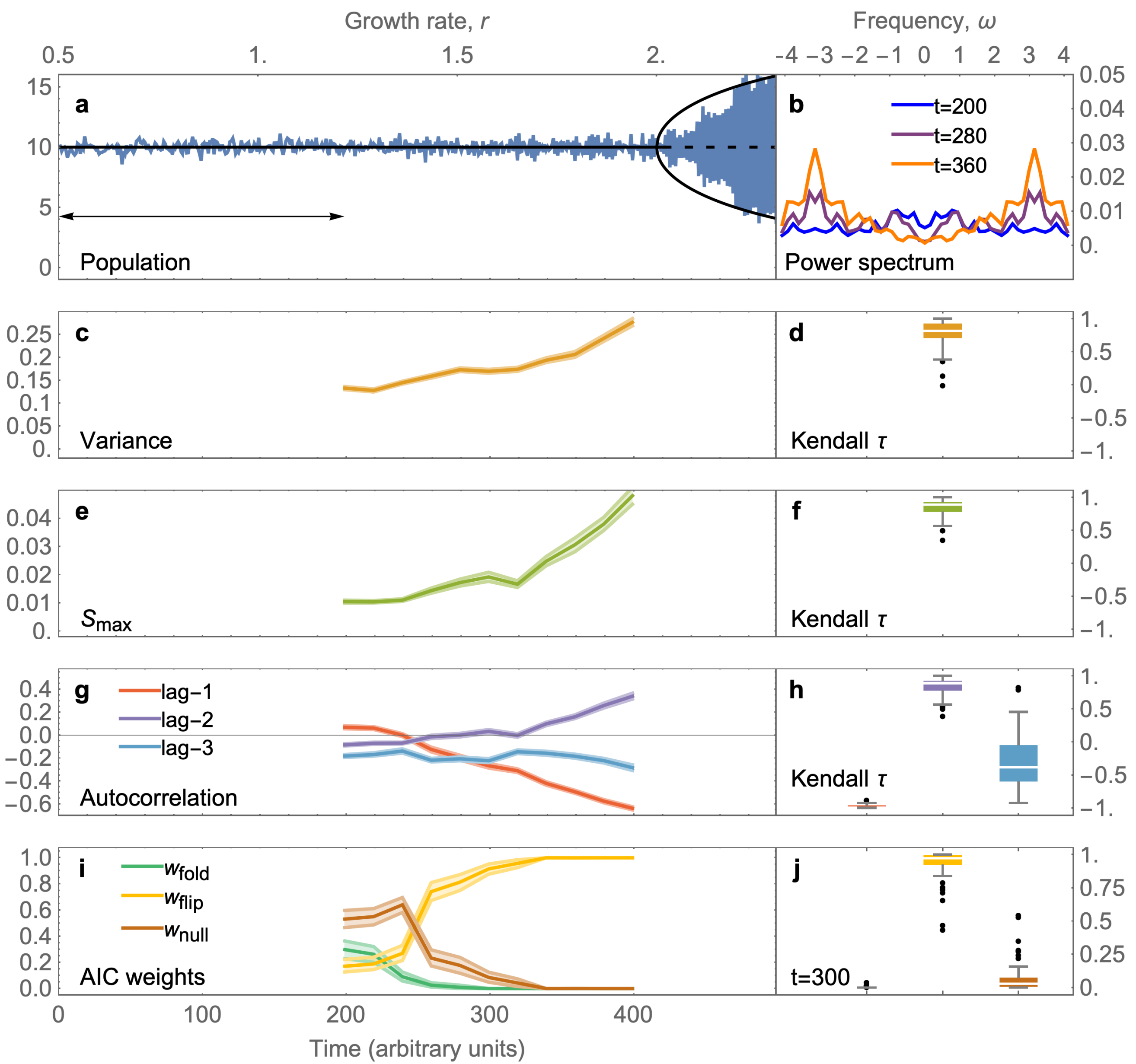
Increasing harvesting rate (F)



Increasing growth rate (r)



- Unimodal power spectrum
- Variance fails to provide a signal
- Smax provides a signal
- AC provides a signal with delayed response at higher lags
- AIC weights faithfully classify bifurcation



- Bimodal power spectrum – dominant frequency π
- Smax provides a slightly stronger signal than variance
- AC trend depends on lag time
- AIC weights faithfully classify bifurcation

Empirical predator-prey system

Fussmann et al. **Crossing the Hopf bifurcation in a live predator-prey system.** Science (2000)

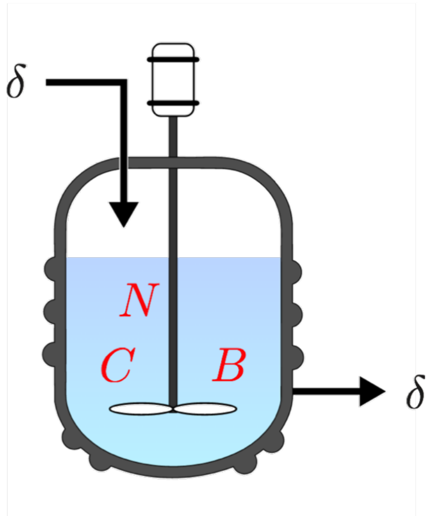
- Chemostat setup with two-species predator prey system
- Run at various dilution rates δ
- Parametrise model with data

$$\frac{dN}{dt} = \delta(N_i - N) - F_C(N)$$

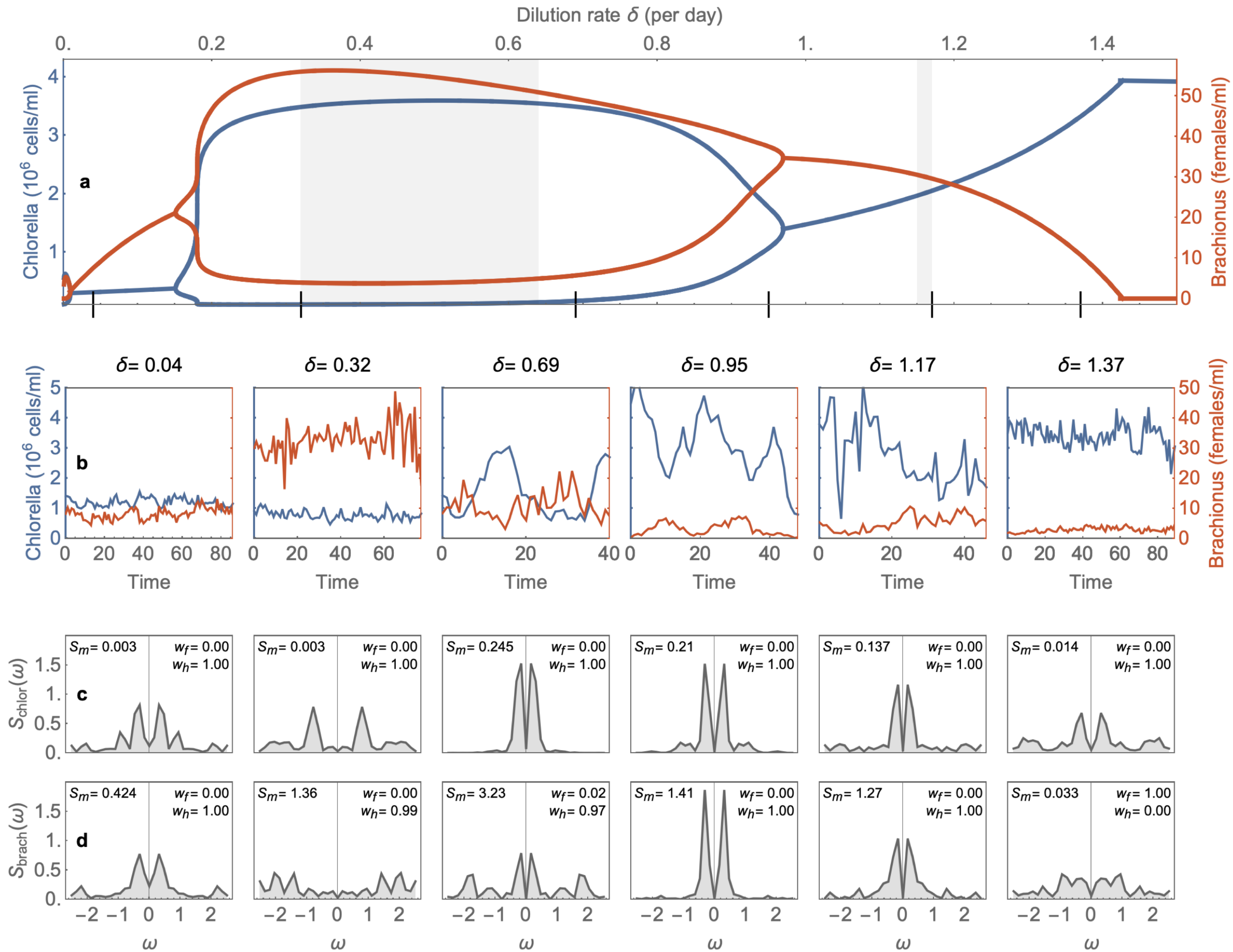
$$\frac{dC}{dt} = F_C(N)C - F_B(C)B/\epsilon - \delta C$$

$$\frac{dR}{dt} = F_B(C)R - (\delta + m + \lambda)R$$

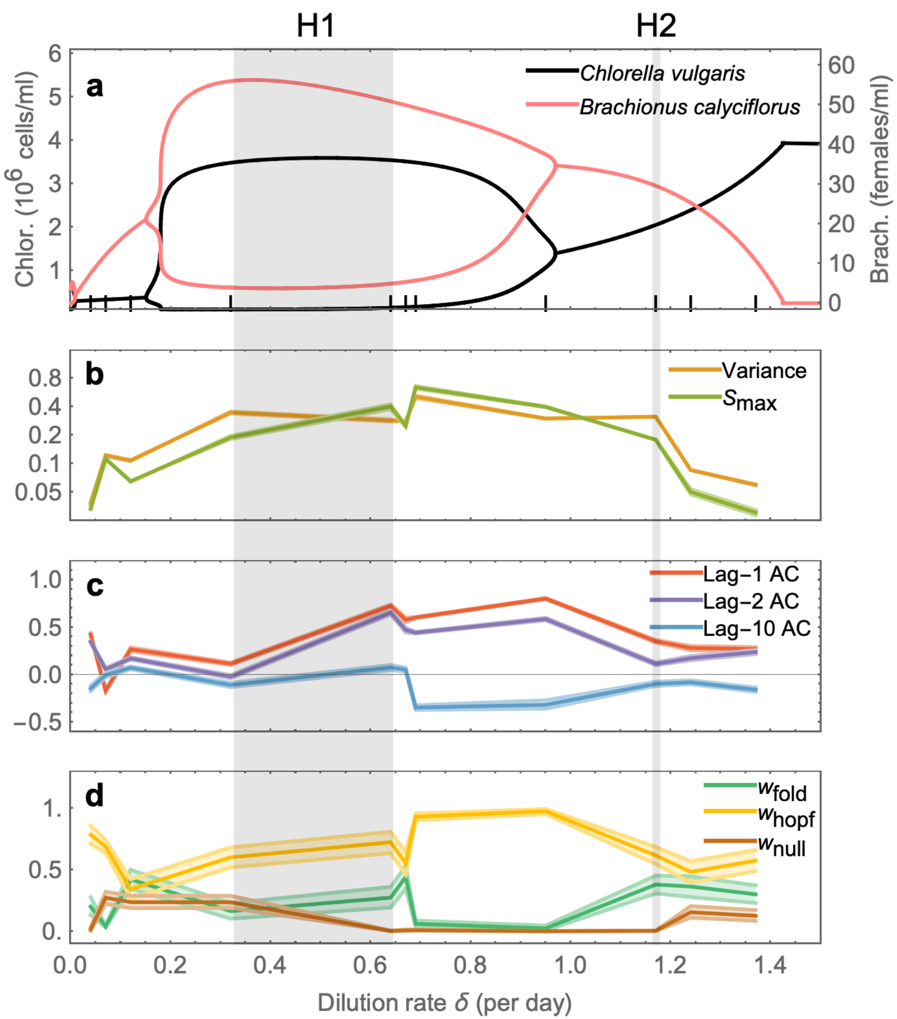
$$\frac{dB}{dt} = F_B(C)R - (\delta + m)B$$



Nitrogen (nutrient)
Chlorella vulgaris (prey)
Brachionus calyciflorus (predator)


Dilution rate δ (per day)
Chlorella (10^6 cells/ml)
Brachionus (females/ml)
a
b
c
d
δ= 0.04
δ= 0.32
δ= 0.69
δ= 0.95
δ= 1.17
δ= 1.37
Time
S_chlor(ω)
S_brach(ω)
ω
S_m= 0.003 w_f= 0.00 w_h= 1.00
S_m= 0.003 w_f= 0.00 w_h= 1.00
S_m= 0.245 w_f= 0.00 w_h= 1.00
S_m= 0.21 w_f= 0.00 w_h= 1.00
S_m= 0.137 w_f= 0.00 w_h= 1.00
S_m= 0.014 w_f= 0.00 w_h= 1.00
S_m= 0.424 w_f= 0.00 w_h= 1.00
S_m= 1.36 w_f= 0.00 w_h= 0.99
S_m= 3.23 w_f= 0.02 w_h= 0.97
S_m= 1.41 w_f= 0.00 w_h= 1.00
S_m= 1.27 w_f= 0.00 w_h= 1.00
S_m= 0.033 w_f= 1.00 w_h= 0.00

Empirical predator-prey system

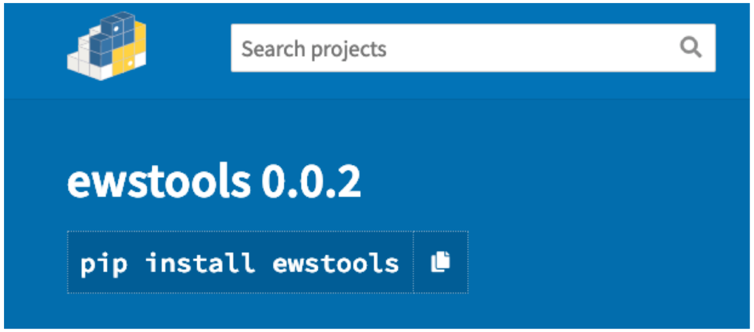


- Smax and variance both provide a signal prior to the bifurcations
- Hopf AIC weight is dominant indicating the type of transition
- Dominant frequency in power spectrum w=1/3 gives T ≈20 days, as observed in oscillatory regime

Conclusions

- Stochasticity offers information – noise can be useful
- Current EWS are not specific to the type of bifurcation that is approaching
- Spectral EWS
    1. provide information on whether the bifurcation is oscillatory (Hopf/Flip/Neimarck-Sacker)
    2. are more sensitive to changes in bifurcation proximity

- All tools developed are available as a Python package.



How can stochasticity be harnessed in your field?

Thank you!