Examples 1:
Inequalities, Exponentials and Logarithms, Inverses

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The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

1 Inequalities

- Practice using set notation
- Know how to manipulate the 'absolute value' sign
- Use graphical interpretation

Example 1.1

Isolate x in the inequality $|3x - 12| < 9$.

$$|3x-12| < 9 \tag{1.1}$$
$$\Rightarrow \quad -9 < 3x - 12 < 9 \tag{1.2}$$
$$\Rightarrow \quad 3 < 3x < 21 \tag{1.3}$$
$$\Rightarrow \quad 1 < x < 7 \tag{1.4}$$

Note 1: We could have divided (1.1) by 3 initially since $\frac{|x|}{a} = \left|\frac{x}{a}\right|$ for $a > 0$. (Slightly faster).

Note 2: $1 < x < 7$ and $x \in (1, 7)$ are equivalent ways of expressing the interval.

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Example 1.2

Describe the set $\{x : |x^2 - 2| > 1\}$ as a union of finite or infinite intervals.

We have either $x^2 - 2 > 1$ or $-(x^2 - 2) > 1$.
The former gives

$$x^2 > 3 \tag{1.5}$$
$$\Rightarrow \quad x > \sqrt{3} \quad \text{or} \quad x < -\sqrt{3} \tag{1.6}$$

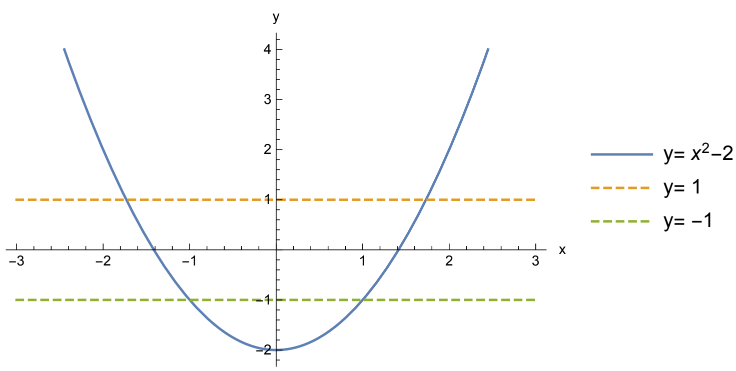
The latter gives

$$x^2 < 1 \tag{1.7}$$
$$\Rightarrow \quad -1 < x < 1 \tag{1.8}$$

Combining the results, we have

$$x \in (-\infty, -\sqrt{3}) \;\cup\; (-1, 1) \;\cup\; (\sqrt{3}, \infty) \tag{1.9}$$

Note: Visualising the problem graphically is a good check / starter:



It is clear that the valid regions lie outside of the dashed lines, agreeing with our answer.

Example 1.3

Find the set of values of x for which $|x-2|-|x+1|<0$

Standard approach: Test values of x in regions bordered by the critical values $x=-1$ and $x=2$.
Case 1: $x \geq 2$

$$x-2-(x+1)<0 \tag{1.10}$$
$$\Rightarrow \quad -3<0 \tag{1.11}$$

This holds for all x in the range $x \geq 2$.

Case 2: $-1 \leq x < 2$

$$-(x-2)-(x+1)<0 \tag{1.12}$$
$$\Rightarrow \quad -2x+1<0 \tag{1.13}$$
$$\Rightarrow \quad x>\frac{1}{2} \tag{1.14}$$

and so the region $\frac{1}{2}<x<2$ is valid.

Case 3: $x \leq -1$

$$-(x-2)+(x+1)<0 \tag{1.15}$$
$$\Rightarrow \quad 3<0 \tag{1.16}$$

which is invalid for all x in this range.

Taking the union of all valid intervals we obtain

$$x \in (1/2, \infty) \tag{1.17}$$

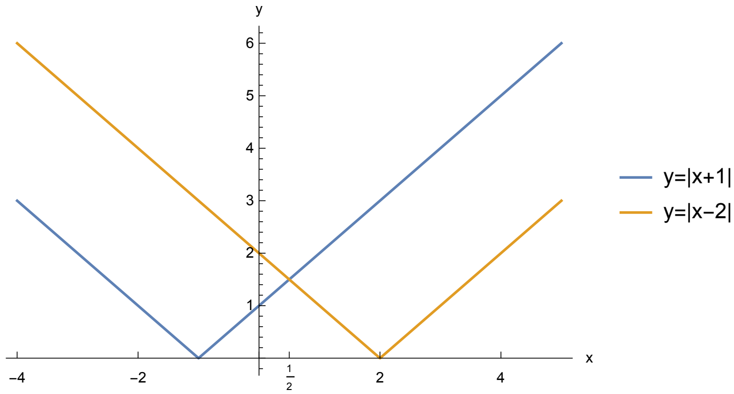
Neat way (optional): Rearrange to

$$|x-2|<|x+1|. \tag{1.18}$$

Since both sides are positive, may take the squares and the inequality still holds. Thus

$$(x-2)^2<(x+1)^2 \tag{1.19}$$
$$\Rightarrow \quad -4x+4<2x+1 \tag{1.20}$$
$$\Rightarrow \quad x>\frac{1}{2} \tag{1.21}$$

Visualise graphically:



For confirmation, we can see the curve $y = |x - 2|$ lies under the curve $y = |x + 1|$ for $x > \frac{1}{2}$.

2 Exponentials and Logarithms

- Practise manipulating exponents
- Familiarise with the laws of exponentials, logarithms and their inverse relation

Example 2.1

Solve the following for x:

$$\sqrt{3^{2x^2+4}} = 27^{x+4} \tag{2.1}$$

Using the laws of exponents, we may simplify the left and right-hand sides as

$$l.h.s = \sqrt{3^{2x^2+4}} = (3^{2x^2+4})^{1/2} = 3^{x^2+2} \tag{2.2}$$

$$r.h.s = 27^{x+4} = (3^3)^{x+4} = 3^{3x+12} \tag{2.3}$$

Note a^x is a one-to-one function, i.e. $a^{x_1} = a^{x_2} \Rightarrow x_1 = x_2$ for any a. Thus we may equate the exponents above to give

$$x^2 + 2 = 3x + 12 \tag{2.4}$$

$$\Rightarrow \quad x^2 - 3x - 10 = 0 \tag{2.5}$$

$$\Rightarrow \quad (x-5)(x+2) = 0 \tag{2.6}$$

$$\Rightarrow \quad x = -2, \;\; 5 \tag{2.7}$$

Example 2.2

Given constants a, b such that $a + b \neq 0$, solve the following for x:

$$a \ln x + b \ln(2x) = 1 \tag{2.8}$$

Using logarithm rules,

$$a \ln x + b \ln(2x) = 1 \tag{2.9}$$

$$\Rightarrow \quad \ln(x^a) + \ln((2x)^b) = 1 \tag{2.10}$$

$$\Rightarrow \quad \ln(x^a(2x)^b) = 1 \tag{2.11}$$

$$\Rightarrow \quad x^a x^b 2^b = e \tag{2.12}$$

$$\Rightarrow \quad x^{a+b} = 2^{-b}e \tag{2.13}$$

$$\Rightarrow \quad x = (2^{-b}e)^{\frac{1}{a+b}} \tag{2.14}$$

Note 1: For $a + b = 0$ equation (2.8) only holds for a single value of b $(= (\ln 2)^{-1})$. Exercise for students?

3 Inverses

- Know how to compute the inverse of basic functions
- Be able to find the domain and range of the inverse function
- Understand that graphically, the inverse function as a reflection through the line $y = x$
- Know requirement for a function to be invertible (one-to-one)

Example 3.1

Find the inverse of the function

$$f(x) = \frac{e^{2x} - 1}{e^{2x} + 1} \tag{3.1}$$

with domain $x \in \mathbb{R}$.

Standard approach is to set $y = f(x)$ and find x in terms of y.

$$\begin{aligned}
& y = \frac{e^{2x} - 1}{e^{2x} + 1} && (3.2)\\
\Rightarrow\quad & (e^{2x} + 1)y = e^{2x} - 1 && (3.3)\\
\Rightarrow\quad & e^{2x}y - e^{2x} = -y - 1 \qquad \text{get } x \text{ terms onto one side} && (3.4)\\
\Rightarrow\quad & e^{2x}(y - 1) = -y - 1 && (3.5)\\
\Rightarrow\quad & e^{2x} = \frac{1 + y}{1 - y} && (3.6)\\
\Rightarrow\quad & x = \frac{1}{2}\ln\left(\frac{1 + y}{1 - y}\right) = f^{-1}(y) && (3.7)
\end{aligned}$$

It doesn't matter which label we choose to demonstrate the function's "rule". Convention is to use x so we may write

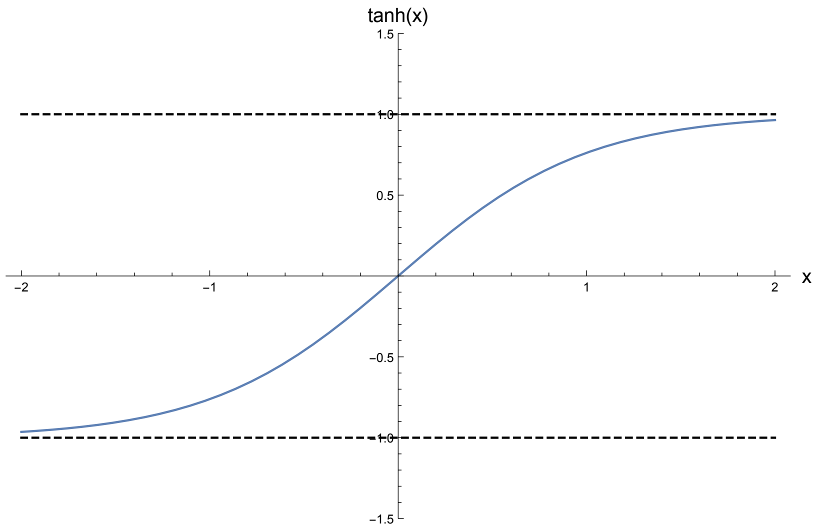
$$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \tag{3.8}$$

Find the domain and range of $f^{-1}(x)$?

The range of f^{-1} is simply the domain of f which is given.

$$\text{Domain}(f) = \text{Range}(f^{-1}) = \mathbb{R} \tag{3.9}$$

The domain of f^{-1} is the range of f. We note that $f(x) = \tanh x$ and use the graph:



See

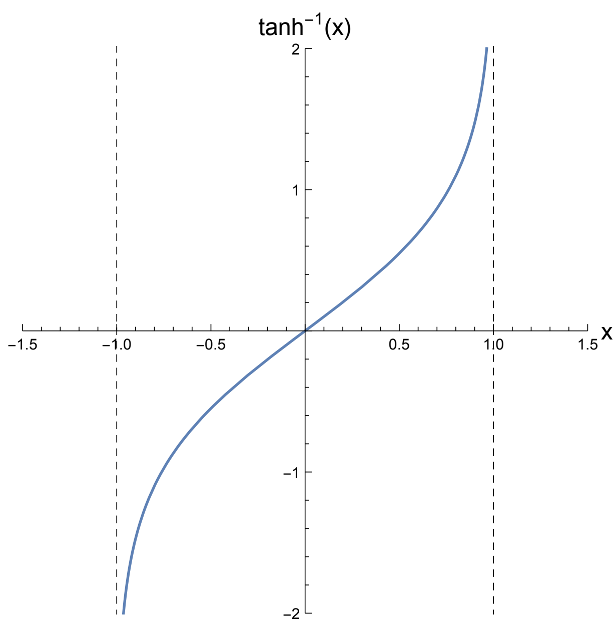
$$Range(f) = Domain(f^{-1}) = (-1, 1) \tag{3.10}$$

A more rigorous approach to finding the range of f would be to show

$$\lim_{x\to-\infty} f(x) = -1, \quad \lim_{x\to\infty} f(x) = 1, \quad f'(x) = \frac{4e^{2x}}{(e^{2x}+1)^2} > 0 \tag{3.11}$$

Sketch $f^{-1}(x) = \tanh^{-1}(x)$

Simply a reflection in the line $y = x$:



Example 3.2

Consider the model for exponential growth of bacteria

$$P(t) = 10e^{2t} \tag{3.12}$$

where P is number of bacterium at time $t \geq 0$ hours.

1. How many bacterium are there initially?
2. At what time does this initial population size double?

1. At time $t = 0$ there are $P(0) = 10$ bacterium.
2. To calculate a time for a given population size, we could find the inverse of (3.12):

   $$t = \frac{1}{2} \ln \left(\frac{P}{10} \right) \tag{3.13}$$

   Then $P = 20$ occurs at $t = \frac{1}{2} \ln 2$ hours.

   Note 1: The function $P(t)$ is one-to-one (exponential) and so we could invert it.

   Note 2: We did not switch the labels for t and P after inverting since they have physical meanings!

   Note 3: Often, we don't bother find the inverse and just substitute a value for P into (3.12) and rearrange (essentially the same procedure).