Examples 2:
Composite Functions, Piecewise Functions, Partial Fractions

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The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

1 Composite Functions

- Compute functional forms of composite functions
- Find their domains
- Even / Odd composites

Example 1.1

Consider the functions $f(x) = \frac{1}{x}$, $x \in (0, \infty)$ and $g(x) = 2x + 1$, $x \in [-2, 2]$.

(a) Determine $f(g(x))$ and its domain

(b) Determine $g(f(x))$ and its domain

(a) Functional form is

$$f(g(x)) = f(2x+1) = \frac{1}{2x+1} \tag{1.1}$$

To be in the domain of $f \circ g$, x must be in the domain of g AND $g(x)$ must be in the domain of f. Thus

$$x \in [-2, 2] \quad \text{and} \quad 2x + 1 \in (0, \infty) \tag{1.2}$$

The second condition gives $x \in (-\frac{1}{2}, \infty)$. Intersected with the first condition, we find the domain of $f \circ g$ to be

$$x \in \left(-\tfrac{1}{2}, 2\right] \tag{1.3}$$

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(b) Functional form is

$$g(f(x)) = g\left(\tfrac{1}{x}\right) = \tfrac{2}{x} + 1 \tag{1.4}$$

- x must be in the domain of f i.e $x \in (0, \infty)$

- $f(x)$ must be in the domain of g i.e $\frac{1}{x} \in [-2, 2]$. So

$$-2 \leq \frac{1}{x} \leq 2 \tag{1.5}$$

$$\Rightarrow \quad \frac{1}{x^2} \leq 4 \tag{1.6}$$

$$\Rightarrow \quad x^2 \geq \frac{1}{4} \tag{1.7}$$

$$\Rightarrow \quad x \geq \frac{1}{2} \quad \text{or} \quad x \leq -\frac{1}{2} \tag{1.8}$$

- Combining sets we get the domain for $g \circ f$ as

$$x \in \left[\tfrac{1}{2}, \infty\right) \tag{1.9}$$

Example 1.2

Suppose $f(x)$ is an even function and $g(x)$ is odd.

(a) Is $f \circ g$ even, odd or neither?

(b) How about $g \circ f$?

Recall definitions and properties of even and odd functions to the class. We have

$$f(-x) = f(x) \quad \text{and} \quad g(-x) = -g(x) \tag{1.10}$$

(a) $f \circ g$ is even since

$$f \circ g(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = f \circ g(x) \tag{1.11}$$

(b) $g \circ f$ is also even since

$$g \circ f(-x) = g(f(-x)) = g(f(x)) = g \circ f(x) \tag{1.12}$$

2 Piecewise Functions

- Rewrite functions from piecewise notation to Heaviside notation and vice versa
- Calculate inverses of (invertible) piecewise functions

Example 2.1

Convert the following function to "piecewise form" and sketch.

$$f(x) = -\frac{1}{x} + H(x+1)\left[\frac{1}{x} - x\right] + H(x)\left[\ln(x+1) + x\right] \tag{2.1}$$

- From the Heaviside arguments we see the function changes form at $x = -1$ and $x = 0$.
- Show that

$$f(x) = \begin{cases} -\frac{1}{x} & x < -1 \\ -x & -1 \leq x < 0 \\ \ln(x+1) & x \geq 0 \end{cases} \tag{2.2}$$

Example 2.2

Write the following using Heaviside notation

$$f(x) = \begin{cases} \cosh x & x < -1 \\ -x^2 + 3 & -1 \leq x < 0 \\ 3x^2 + 1 & 0 \leq x < 1 \\ \sin x & x \geq 1 \end{cases} \tag{2.3}$$

- We note that the function changes form at $x = -1, 0, 1$. Therefore we require the Heaviside functions $H(x+1)$, $H(x)$ and $H(x-1)$ respectively.
- Begin with $f(x) = \cosh x + \ldots$
- At $x = -1$ the function transforms. Get rid of $\cosh x$ and add $(-x^2 + 3)$.

$$f(x) = \cosh x + H(x+1)\left(-\cosh x - x^2 + 3\right) + \ldots \tag{2.4}$$

- At $x = 0$ get rid of $(-x^2 + 3)$ and add $3x^2 + 1$:

$$f(x) = \cosh x + H(x+1)\left(-x^2 + 3 - \cosh x\right) + H(x)\left(-(-x^2 + 3) + 3x^2 + 1\right) + \ldots \tag{2.5}$$

- Finally, at $x = 1$ get rid of $(3x^2 + 1)$ and add $\sin x$:

$$f(x) = \cosh x + H(x+1)\left(-x^2 + 3 - \cosh x\right) + H(x)\left(3x^2 + 1 - (-x^2 + 3)\right) + H(x-1)\left(-(3x^2 + 1) + \sin x\right) \tag{2.6}$$

$$= \cosh x + H(x+1)\left(-x^2 + 3 - \cosh x\right) + H(x)\left(4x^2 - 2)\right) + H(x-1)\left(-3x^2 - 1 + \sin x\right) \tag{2.7}$$

Example 2.3 - *one for the students (they'll need waking up by now)*

Express the following function using Heaviside notation

$$f(x) = \begin{cases} e^{-x} & x < 1 \\ \ln(2x) & 1 \leq x < \sqrt{2} \\ \ln(\sqrt{x}) & x \geq \sqrt{2} \end{cases} \tag{2.8}$$

Solution:

$$f(x) = e^{-x} + H(x-1)\left(\ln(2x) - e^{-x}\right) + H(x - \sqrt{2})\ln\left(\frac{1}{2\sqrt{x}}\right) \tag{2.9}$$

Example 2.4

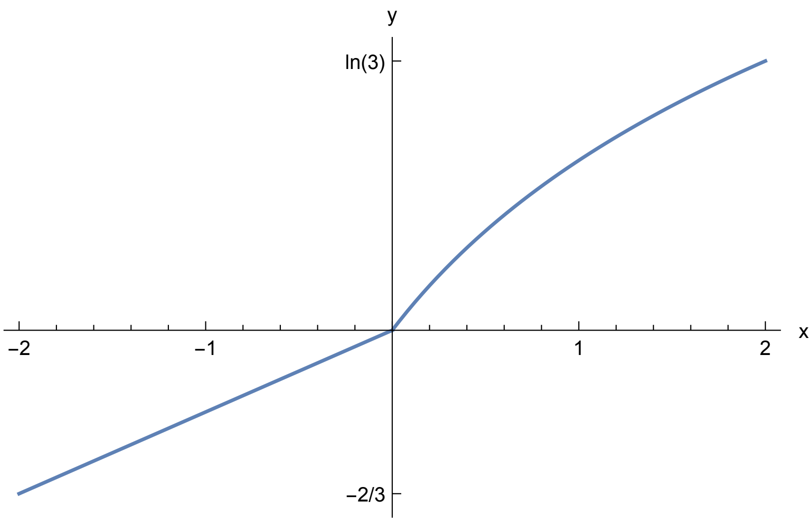
Consider the following piecewise-defined function

$$f(x) = \begin{cases} x/3 & -2 \leq x \leq 0 \\ \ln(x+1) & 0 \leq x \leq 2 \end{cases} \tag{2.10}$$

(a) Sketch the function

(b) Find its inverse

(a) Plot



(b) Since f is one-to-one, we may invert it. Consider each piece separately:

- For $-2 \leq x \leq 0$ we have $-2/3 \leq y \leq 0$. Inverting gives $x = 3y$.
- For $0 \leq x < 2$ we have $0 \leq y \leq \ln 3$. Inverting gives $x = e^y - 1$.
- Put the two together...

$$f^{-1}(y) = \begin{cases} 3y & -2/3 \leq y \leq 0 \\ e^y - 1 & 0 \leq y \leq \ln 3 \end{cases} \tag{2.11}$$

3 Partial Fractions

- Express proper rational functions in terms of partial fractions
- Reduce improper rational functions to a polynomial and a proper rational function (long division)

Example 3.1†

Express the following proper rational functions as partial fractions.

(a)

$$\frac{x+23}{x^2+x-6} \tag{3.1}$$

(b)

$$\frac{5x^2+3x-5}{x^3+x^2-2x-2} \tag{3.2}$$

(c)

$$\frac{1}{x(x-1)^3} \tag{3.3}$$

(a)

$$\begin{aligned} \frac{x+23}{x^2+x-6} &= \frac{x+23}{(x+3)(x-2)} && \text{factorize the denominator} && (3.4)\\ &= \frac{A}{x+3}+\frac{B}{x-2} && \text{partial fraction form for linear denominators} && (3.5)\\ &= \frac{A(x-2)+B(x-3)}{(x+3)(x-2)} && \text{put over a common denominator} && (3.6) \end{aligned}$$

- Equate the numerators to get

$$A(x-2)+B(x+3)=x+23 \tag{3.7}$$

- **Now either**: Group terms together and equate coefficients,

$$(A+B)x-2A+3B=x+23 \tag{3.8}$$

to give simultaneous equations

$$A+B=1 \tag{3.9}$$

$$3B-2A=23 \tag{3.10}$$

which solve to give $A=-4$, $B=5$.

†Note: this is very useful for integration

- **Or**: Set values of x to simplify (3.7) (we may do this since this equation must hold for all x). Setting $x = 2$ gives

$$5B = 25 \tag{3.11}$$
$$\Rightarrow \quad B = 5. \tag{3.12}$$

Setting $x = -3$ gives

$$-5A = 20 \tag{3.13}$$
$$\Rightarrow \quad A = -4 \tag{3.14}$$

- The decomposition is then

$$\frac{x+23}{x^2+x-6} = \frac{5}{x-2} - \frac{4}{x+3} \tag{3.15}$$

(b) - We first need to factorize the denominator of

$$\frac{5x^2+3x-5}{x^3+x^2-2x-2} \tag{3.16}$$

- Since cubic, we guess a root using trial and error from the factors of -2. Find $x = -1$ is a root, thus $(x+1)$ is a factor.
- Use long division to find the other factor:

$$\begin{array}{r|l} & x^2 \qquad\quad - 2 \\ \hline x+1 & x^3 + x^2 - 2x - 2 \\ & -x^3 - x^2 \\ \hline & \qquad\quad -2x - 2 \\ & \qquad\quad\;\; 2x + 2 \\ \hline & \qquad\qquad\quad 0 \end{array} \tag{3.17}$$

and so

$$x^3 + x^2 - 2x - 2 = (x+1)(x^2-2) \tag{3.18}$$

- Since we have a linear and a quadratic term, the partial fraction decomposition takes the form

$$\frac{5x^2+3x-5}{(x+1)(x^2-2)} = \frac{A}{x+1} + \frac{Bx+c}{x^2-2} \tag{3.19}$$

- Make the common denominator and equate the numerators to give

$$A(x^2-2) + (Bx+C)(x+1) = 5x^2+3x-5 \tag{3.20}$$

- Group terms:

$$(A+B)x^2 + (B+C)x + C - 2A = 5x^2 + 3x - 5 \tag{3.21}$$

- Solve the system of equations

$$A + B = 5 \tag{3.22}$$
$$B + C = 3 \tag{3.23}$$
$$C - 2A = -5 \tag{3.24}$$

to get $(A, B, C) = (3, 2, 1)$. Therefore

$$\frac{5x^2 + 3x - 5}{x^3 + x^2 - 2x - 2} = \frac{3}{x+1} + \frac{2x+1}{x^2 - 2} \tag{3.25}$$

Note: We could have used a combo of both methods here; setting $x = -1$ in (3.19) would have immediately given us $A = 3$ and saved us a bit of work.

(c) - Here the denominator has a linear term and a cubic term with a 3-fold repeated root. This has decomposition

$$\frac{1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \tag{3.26}$$

- Forming the common denominator and equating numerators gives

$$A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx = 1 \tag{3.27}$$

- Setting $x = 1$ and $x = 0$ separately immediately gives $D = 1$ and $A = -1$.
- Grouping terms, we have

$$(A+B)x^3 + (-3A - 2B + C)x^2 + (3A + B - C + D)x - A = 1 \tag{3.28}$$

- The cubic component gives

$$A + B = 0 \quad \Rightarrow \quad B = -A = 1 \tag{3.29}$$

- The quadratic component gives

$$-3A - 2B + C = 0 \quad \Rightarrow \quad C = 3A + 2B = -1 \tag{3.30}$$

- Finally,

$$\frac{1}{x(x-1)^3} = -\frac{1}{x} + \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} \tag{3.31}$$

Example 3.1

Decompose the following improper rational function into its partial fractions:

$$\frac{x^4 + 4x^3 - 5x^2 - 15x + 14}{x^2 + 2x - 8} \tag{3.32}$$

- Since this rational function is *improper* we first need to break it down into a polynomial and a *proper* rational function. Using long division,

$$\begin{array}{r|l} & x^2 \quad + 2x \quad - 1 \\ \hline x^2 + 2x - 8 & x^4 + 4x^3 - 5x^2 - 15x + 14 \\ & -x^4 - 2x^3 + 8x^2 \\ \hline & 2x^3 + 3x^2 - 15x \\ & -2x^3 - 4x^2 + 16x \\ \hline & -x^2 \quad + x + 14 \\ & x^2 \quad + 2x \quad - 8 \\ \hline & 3x \quad + 6 \end{array} \tag{3.33}$$

and so

$$\frac{x^4 + 4x^3 - 5x^2 - 15x + 14}{x^2 + 2x - 8} = x^2 + 2x - 1 + \frac{3x + 6}{x^2 + 2x - 8} \tag{3.34}$$

- The rational part may be written as

$$\frac{3x + 6}{(x + 4)(x - 2)} = \frac{A}{x + 4} + \frac{B}{x - 2} \tag{3.35}$$

giving

$$A(x - 2) + B(x + 4) = 3x + 6 \tag{3.36}$$

- Setting $x = 2$ gives $6B = 12 \Rightarrow B = 2$.

- Setting $x = -4$ gives $-6A = -6 \Rightarrow A = 1$.

- Altogether

$$\frac{x^4 + 4x^3 - 5x^2 - 15x + 14}{x^2 + 2x - 8} = x^2 + 2x - 1 + \frac{1}{x + 4} + \frac{2}{x - 2} \tag{3.37}$$