# Examples 3:
# Trigonometric Functions, Hyperbolic Functions

October 3, 2016

The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

## 1 Trigonometric Functions

- Learn / derive essential trig values
- Learn and apply compound-angle formulae
- Practise manipulating trig identities to solve problems
- Find amplitude and phase of superposition of waves (same freq.)

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**Example 1.1**

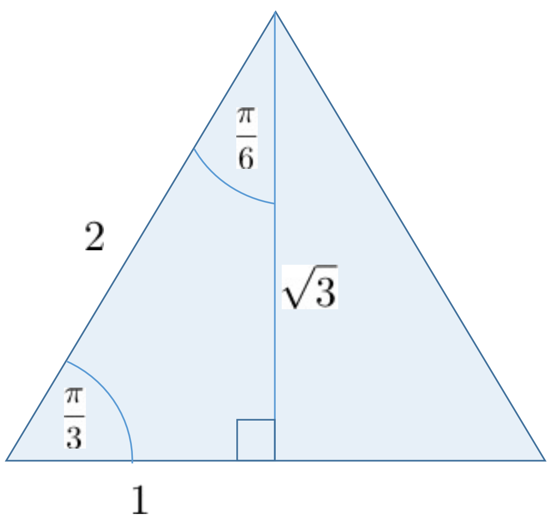
In drawing the appropriate right-angled triangles, find the exact values of

(a) $\sin\left(\frac{\pi}{3}\right)$

(b) $\cos\left(\frac{\pi}{4}\right)$

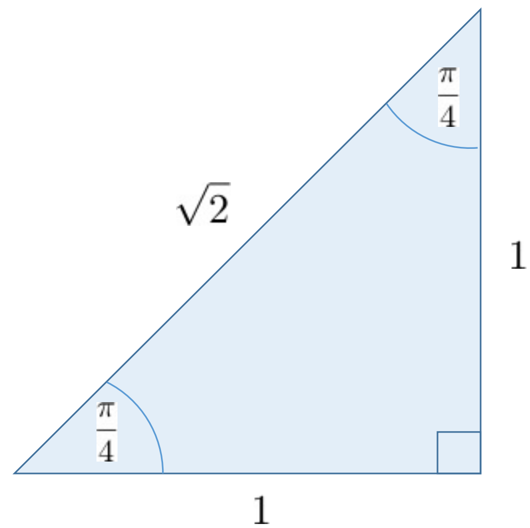
($a$) Trig values of angles $\frac{\pi}{6}$ and $\frac{\pi}{3}$ may be found using one half of an equilateral triangle. From this we can read off

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \tag{1.1}$$




($b$) Trig values for $\frac{\pi}{4}$ may be found from a right-angled isosceles triangle. From this we see

$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{1.2}$$



**Example 1.2**

Using the compound angle formulae, find the exact values of / expressions for

(a) $\cos\left(\frac{2\pi}{3}\right)$

(b) $\sin\left(\frac{5\pi}{12}\right)$

(c) $\cos\left(\frac{\pi}{2}-\theta\right)$

(d) $\sin\left(\theta+\pi\right)$

(a) By the cosine compound-angle formula (or double angle formula), we have

$$\begin{aligned}\cos\left(\tfrac{2\pi}{3}\right) = \cos\left(\tfrac{\pi}{3}+\tfrac{\pi}{3}\right) &= \cos\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{\pi}{3}\right) - \sin\left(\tfrac{\pi}{3}\right)\sin\left(\tfrac{\pi}{3}\right) && (1.3)\\ &= (1/2)^2 - (\sqrt{3}/2)^2 && (1.4)\\ &= -1/2 && (1.5)\end{aligned}$$

(b) We may write

$$\begin{aligned}\sin\left(\tfrac{5\pi}{12}\right) = \sin\left(\tfrac{\pi}{4}+\tfrac{\pi}{6}\right) &= \sin\left(\tfrac{\pi}{4}\right)\cos\left(\tfrac{\pi}{6}\right) + \cos\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{\pi}{6}\right) && (1.6)\\ &= (1/\sqrt{2})(\sqrt{3}/2) + (1/\sqrt{2})(1/2) && (1.7)\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} && (1.8)\end{aligned}$$

(c) We have

$$\begin{aligned}\cos\left(\tfrac{\pi}{2}-\theta\right) &= \cos\left(\tfrac{\pi}{2}\right)\cos\theta + \sin\left(\tfrac{\pi}{2}\right)\sin\theta && (1.9)\\ &= \sin\theta && (1.10)\end{aligned}$$

As well, $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$. These relations can be easily seen from a right-angled triangle.

(d) Finally,

$$\begin{aligned}\sin(\theta+\pi) &= \sin\theta\cos\pi + \cos\theta\sin\pi && (1.11)\\ &= -\sin\theta && (1.12)\end{aligned}$$

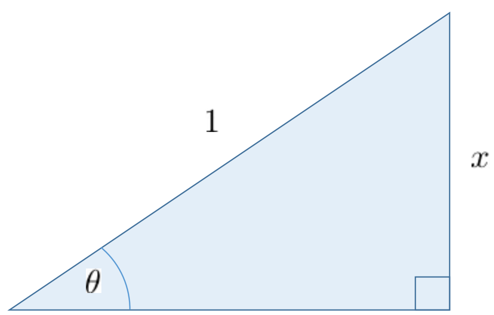
**Example 1.3**

Using right-angled triangles or trig identities, find an expression for

$$f(x) = \tan\left(\sin^{-1}(x)\right) \qquad x \in (-1, 1) \tag{1.13}$$

- Let $\theta = \sin^{-1}(x)$. Then $\sin\theta = \sin(\sin^{-1}(x)) = x$, which holds for all $x \in [-1, 1]$
- We wish to find $\tan\theta$. We know $\sin\theta$ in terms of $x$. We may do either of the following:

1. Draw the relevant right-angled triangle with unit hypotenuse as below.




The missing side may be calculated using Pythagoras' Theorem, giving $\sqrt{1-x^2}$. We can then read off $\tan\theta$ as

$$\tan\theta = \frac{x}{\sqrt{1-x^2}} \tag{1.14}$$

2. Use trigonometric identities to write $\tan\theta$ in terms of $\sin\theta$. We have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \tag{1.15}$$

Since $x = \sin\theta$, we get the result as before.

- Either method yields

$$f(x) = \frac{x}{\sqrt{1-x^2}} \tag{1.16}$$

**Example 1.4**

Find all values of $x \in [0, 2\pi)$ that satisfy

$$3\sin x - \sin 3x + \cos 2x - 1 = 0 \tag{1.17}$$

- We need to reduce the trig terms to the same form. We will reduce to $\sin x$ since it's already in the equation.
- Start with $\sin 3x$:

$$\begin{aligned} \sin 3x = \sin(2x + x) &= \sin 2x \cos x + \cos 2x \sin x && (1.18)\\ &= (2\sin x \cos x)\cos x + (1 - 2\sin^2 x)\sin x && (1.19)\\ &= 2\sin x(1 - \sin^2 x) + (1 - 2\sin^2 x)\sin x && (1.20)\\ &= 3\sin x - 4\sin^3 x && (1.21) \end{aligned}$$

- And we know

$$\cos 2x = 1 - 2\sin^2 x \tag{1.22}$$

- Then equation (1.17) may be written

$$\begin{aligned} 3\sin x - (3\sin x - 4\sin^3 x) + (1 - 2\sin^2 x) - 1 = 0 && (1.23)\\ \Rightarrow 4\sin^3 x - 2\sin^2 x = 0 && (1.24)\\ \Rightarrow \sin^2 x(2\sin x - 1) = 0 && (1.25) \end{aligned}$$

- Either $\sin x = 0$ giving $x = 0, \pi$.
- Or $\sin x = \frac{1}{2}$ giving $x = \frac{\pi}{6}, \frac{5\pi}{6}$.
- Combining all possible solutions we have

$$x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi \tag{1.26}$$

**Example 1.5**

Express the function

$$f(t) = 3\sin 2t - 4\cos 2t \tag{1.27}$$

in the form $A\sin(\omega t + \phi)$.

- The frequency can be read off as $\omega = 2$. Expanding out the required form, we have

$$A\sin(2t+\phi) = A\sin 2t\cos\phi + A\cos 2t\sin\phi \tag{1.28}$$
$$= (A\cos\phi)\sin 2t + (A\sin\phi)\cos 2t \tag{1.29}$$

- Comparing coefficients with $f(t)$ we require

$$A\cos\phi = 3 \tag{1.30}$$
$$A\sin\phi = -4 \tag{1.31}$$

- Summing the squares of these two expressions gives

$$A^2\cos^2\phi + A^2\sin^2\phi = 3^2 + (-4)^2 \tag{1.32}$$
$$\Rightarrow A^2(\cos^2\phi + \sin^2\phi) = 25 \tag{1.33}$$
$$\Rightarrow A = 5 \tag{1.34}$$

  By convention, we take the positive root since $A$ represents the amplitude. Taking the negative root is also acceptable however, just note we would get a different phase $\phi$ to compensate.

- Dividing the expressions gives

$$\tan\phi = -\frac{4}{3} \tag{1.35}$$

- Check which quadrant $\phi$ lies in. Since $A > 0$ we must have $\cos\phi > 0, \sin\phi < 0$, putting $\phi$ in the fourth quadrant.

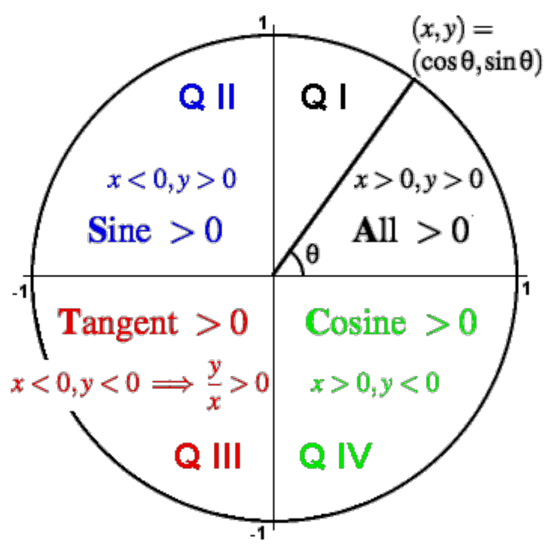
- Since $\tan^{-1}\left(-\frac{4}{3}\right) \in [-\frac{\pi}{2}, 0]$ (can see this by drawing a quick graph of tan), this belongs to the fourth quadrant and so [a]

$$\phi = \tan^{-1}\left(-\tfrac{4}{3}\right) \tag{1.36}$$

- Putting this together gives

$$f(t) = 5\sin(2t + \tan^{-1}\left(-\tfrac{4}{3}\right)) \tag{1.37}$$

[a] if the quadrants do not match, add $\pi$ to get the correct phase (tan is $\pi$-periodic)



**Example 1.6** - extra practice

Express the function

$$f(t) = -5\sin\pi t + 2\cos\pi t \tag{1.38}$$

in the form $A\sin(\omega t + \phi)$.

- Get

$$A\sin\phi = 2 \tag{1.39}$$

$$A\cos\phi = -5 \tag{1.40}$$

- Amplitude

$$A = \sqrt{2^2 + 5^2} = \sqrt{29} \tag{1.41}$$

- Phase

$$\tan\phi = -\frac{2}{5} \tag{1.42}$$

- $\phi$ lies in the second quadrant (sin>0, cos<0)
- $\tan^{-1}(-2/5)$ lies in the fourth quadrant and so

$$\phi = \tan^{-1}\left(-\frac{2}{5}\right) + \pi \tag{1.43}$$

- Full solution

$$f(t) = \sqrt{29}\sin\left(\pi t + \tan^{-1}\left(-\frac{2}{5}\right) + \pi\right) \tag{1.44}$$

## 2 Hyperbolic Functions

- Obtain explicit forms for the inverse hyperbolic functions
- Practise deriving identities and manipulating expressions

### Example 2.1

Find the inverse of $f(x) = \sinh x$

- Write $y = f(x)$ and then find $x$ in terms of $y$:

$$y = \frac{e^x - e^{-x}}{2} \tag{2.1}$$
$$\Rightarrow \quad e^x - 2y - e^{-x} = 0 \tag{2.2}$$
$$\Rightarrow \quad e^{2x} - 2ye^x - 1 = 0 \tag{2.3}$$
$$\Rightarrow \quad e^x = \frac{1}{2}\left(2y + \sqrt{4y^2 + 4}\right) \quad \text{positive root since } e^x > 0 \tag{2.4}$$
$$\Rightarrow \quad e^x = y + \sqrt{y^2 + 1} \tag{2.5}$$
$$\Rightarrow \quad x = \ln(y + \sqrt{y^2 + 1}) \tag{2.6}$$

- We deduce that

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) \tag{2.7}$$

Hence find the exact value of $x$ that satisfies $\sinh x = 1$ *in terms of logarithms*

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$$\sinh x = 2 \quad \Rightarrow \quad x = \sinh^{-1}(2) = \ln(2 + \sqrt{5}) \tag{2.8}$$

**Example 2.2**

(a) Prove that $\sinh(2x) = 2\sinh(x)\cosh(x)$

(b) Find all values of $x$ that satisfy

$$\sinh(2x) - 3\tanh(x) - \sinh(x) = 0 \tag{2.9}$$

(a) Starting with the r.h.s:

$$2\sinh x\cosh x = 2\left(\frac{1}{4}(e^x - e^{-x})(e^x + e^{-x})\right) \tag{2.10}$$

$$= \frac{1}{2}(e^{2x} - e^{-2x}) \tag{2.11}$$

$$= \sinh(2x) \tag{2.12}$$

(b) We have

$$\sinh(2x) - 3\tanh x - \sinh x = 0 \tag{2.13}$$

$$\Rightarrow \quad 2\sinh x\cosh x - \frac{3\sinh x}{\cosh x} - \sinh x = 0 \tag{2.14}$$

$$\Rightarrow \quad 2\sinh x\cosh^2(x) - 3\sinh x - \sinh x\cosh x = 0 \qquad \text{multiply through by } \cosh x \tag{2.15}$$

$$\Rightarrow \sinh x(2\cosh^2(x) - \cosh x - 3) = 0 \tag{2.16}$$

$$\Rightarrow \sinh x(2\cosh x - 3)(\cosh x + 1) = 0 \tag{2.17}$$

Possibilities are:

- $\sinh x = 0$, i.e. $x = 0$.
- $\cosh x = 3/2$ i.e $x = \cosh^{-1}(3/2) = \ln\left(\frac{1}{2}(3+\sqrt{5})\right)$ *

Note that $\cosh x = -1$ has no solutions since $\cosh x \geq 1$ for all $x$.

*using $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$