Examples 4:
Limits and Continuity

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The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

1 Limits

- Solve limit problems using standard limit rules.
- Solve limit problems using the definition of a limit
- Practise applying the Squeeze Theorem
- Investigate existence of limits for piecewise functions

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Example 1.1 - true / false statements

Warm up with the following true / false statements

(a) If f is continuous at a,

$$\lim_{x \to a} f(x) = f(a) \tag{1.1}$$

(b) The limit

$$\lim_{n \to \infty} \frac{1}{n^p} = 0 \tag{1.2}$$

for any $p \in \mathbb{R}$.

(c) The limit

$$\lim_{x \to \infty} \left(\sin^2(\sqrt{x}) + \cos^2(\sqrt{x}) \right) \tag{1.3}$$

does not exist.

(d) Suppose

$$g(x), h(x) \to 5 \quad \text{as} \quad x \to \infty \tag{1.4}$$

$$\text{and} \qquad g(x) \le f(x) \le h(x) \tag{1.5}$$

Then

$$\lim_{x \to \infty} f(x) = 5 \tag{1.6}$$

(a) *True.* This is very useful, since it tells us that when evaluating the limit of a function *at a point where it is continuous*, we may just plug the value in. For example

$$\lim_{x \to 2} \frac{x^2 + 3}{x - 1} = \frac{4 + 3}{2 - 1} = 7 \tag{1.7}$$

(b) *False.* This limit is only satisfied for $p > 0$. Note that

$$\lim_{n \to \infty} \frac{1}{n^p} = \begin{cases} 0 & p > 0 \\ 1 & p = 0 \\ \infty & p < 0 \end{cases} \tag{1.8}$$

(c) *False.* Since

$$\sin^2(f(x)) + \cos^2(f(x)) = 1 \tag{1.9}$$

for any function $f(x)$. Any limit of a constant is just itself, so in this case the limit is 1.

(d) *True.* This is an example of the Squeeze Theorem.

Example 1.2 - Limits with an "indeterminate form"

Evaluate the following limits

(a)
$$\lim_{x\to 3} \frac{x^2-2x-3}{x-3} \tag{1.10}$$

(b)
$$\lim_{x\to\infty} \sqrt{x^2+2x}-x \tag{1.11}$$

(a) Upon substituting 3 into the expression we see this limit has the indeterminate form "0/0". Rewriting the function, we have

$$\frac{x^2-2x-3}{x-3} = \frac{(x-3)(x+1)}{x-3} \tag{1.12}$$
$$= x+1 \qquad \text{provided that } x \neq 3 \tag{1.13}$$

Note in taking the limit we get arbitrarily close to $x = 3$ but never actually attain it, hence we may cancel the factors of $(x-3)$. Finally,

$$\lim_{x\to 3} x+1 = 4 \tag{1.14}$$

(b) This has the indeterminate form "$\infty - \infty$". We get around this by converting the expression into a ratio:

$$\sqrt{x^2+2x}-x = \frac{(\sqrt{x^2+2x}-x)(\sqrt{x^2+2x}+x)}{\sqrt{x^2+2x}+x} \tag{1.15}$$
$$= \frac{2x}{\sqrt{x^2+2x}+x} \qquad \text{multiply out numerator} \tag{1.16}$$
$$= \frac{2}{\sqrt{1+\frac{2}{x}}+1} \qquad \text{divide by highest power} \tag{1.17}$$

Now the limit is fairly simple

$$\lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{2}{x}}+1} = \frac{2}{\sqrt{1}+1} = 1 \tag{1.18}$$

Example 1.3 - Application of the Squeeze Theorem

Evaluate the following limits

(a)
$$\lim_{x\to\infty} e^{-0.1x}\sin x \tag{1.19}$$

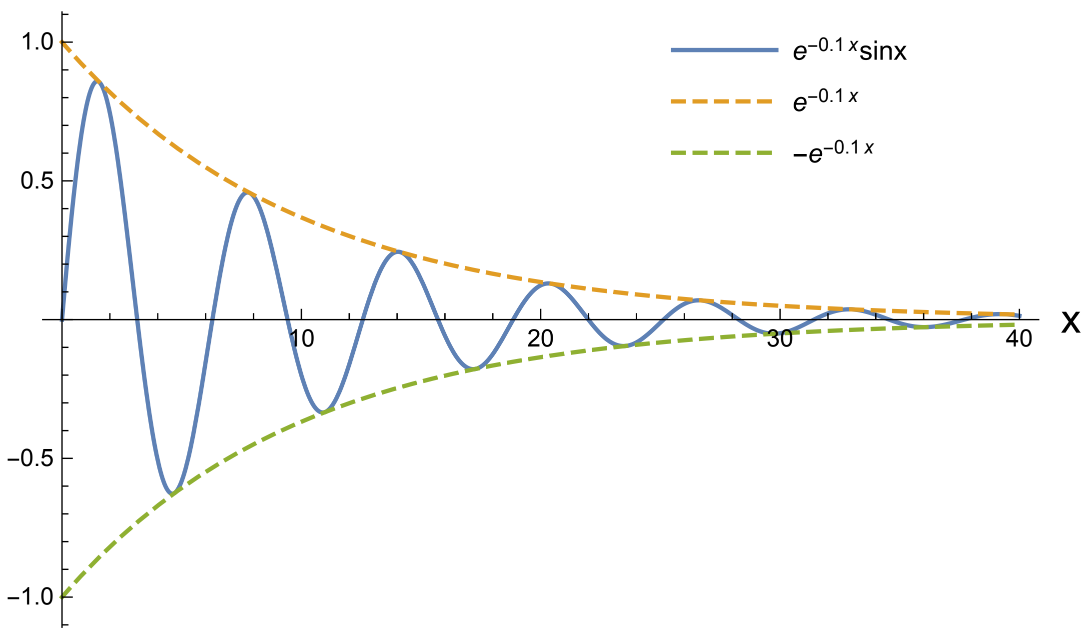
(b)
$$\lim_{x\to 0} x^2 e^{\cos\left(\frac{1}{x}\right)} \tag{1.20}$$

(a) We use the fact that $\sin x$ is bounded:

$$-1 \le \sin x \le 1 \tag{1.21}$$
$$\Rightarrow \quad -e^{-0.1x} \le e^{-0.1x}\sin x \le e^{-0.1x} \qquad \text{since } e^{-0.1x} > 0 \tag{1.22}$$

Now
$$\lim_{x\to\infty} -e^{-0.1x} = \lim_{x\to\infty} e^{-0.1x} = 0 \tag{1.23}$$
and so by the Squeeze Theorem we must have
$$\lim_{x\to\infty} e^{-0.1x}\sin x = 0 \tag{1.24}$$

(This represents an oscillation with exponentially decaying amplitude)



(b) Now use the fact that $\cos(1/x)$ is bounded:

$$-1 \leq \cos(1/x) \leq 1 \tag{1.25}$$

$$\Rightarrow \quad e^{-1} \leq e^{\cos(\frac{1}{x})} \leq e \qquad \text{(since } e^x \text{ is an increasing function we may do this)} \tag{1.26}$$

$$\Rightarrow \quad x^2 e^{-1} \leq x^2 e^{\cos(\frac{1}{x})} \leq x^2 e \tag{1.27}$$

The outside limits are

$$\lim_{x \to 0} x^2 e^{-1} = \lim_{x \to 0} x^2 e = 0 \tag{1.28}$$

and so by the Squeeze Theorem,

$$\lim_{x \to 0} x^2 e^{\cos(\frac{1}{x})} = 0 \tag{1.29}$$

Example 1.4 - Limits from first principles

Prove the following, using the definition of a limit

(a)

$$\lim_{n\to\infty} \frac{n^2+1}{n^2-1} = 1 \tag{1.30}$$

(b)

$$\lim_{x\to 2}(3x-1) = 5 \tag{1.31}$$

(a) We need to show that for any $\epsilon > 0$, we can find an N such that

$$n > N \quad \Rightarrow \quad \left|\frac{n^2+1}{n^2-1} - 1\right| < \epsilon. \tag{1.32}$$

Investigating the condition further we see that we require

$$\left|\frac{n^2+1-(n^2-1)}{n^2-1}\right| < \epsilon \tag{1.33}$$

$$\Rightarrow \quad \left|\frac{2}{n^2-1}\right| < \epsilon \tag{1.34}$$

Since we are interested in the limit as $n \to \infty$ it is reasonable to assume that $n > 1$. Thus we may drop the absolute value signs which gives

$$n^2 - 1 > \frac{2}{\epsilon} \tag{1.35}$$

$$\Rightarrow \quad n > \sqrt{1+\frac{2}{\epsilon}} \quad \text{taking the positive root since } n > 1 \tag{1.36}$$

Now set $N = \sqrt{1+2/\epsilon}$ meaning that for $n > N$ we have

$$\left|\frac{n^2+1}{n^2-1} - 1\right| < \epsilon \tag{1.37}$$

proving the assumed limit of 1 is correct.

(b) We must show that for any $\epsilon > 0$ we can find a δ such that

$$|x-2| < \delta \quad \Rightarrow \quad |(3x-1)-5| < \epsilon. \tag{1.38}$$

Investigating the condition further, see that require

$$|3x-6| < \epsilon \tag{1.39}$$
$$\Rightarrow \quad |x-2| < \epsilon/3 \tag{1.40}$$

So if we set $\delta = \epsilon/3$ we have

$$|x-2| < \delta \tag{1.41}$$
$$\Rightarrow |x-2| < \epsilon/3 \tag{1.42}$$
$$\Rightarrow |3x-6| < \epsilon \tag{1.43}$$
$$\Rightarrow |(3x-1)-5| < \epsilon \tag{1.44}$$

as required to prove

$$\lim_{x\to 2}(3x-1) = 5 \tag{1.45}$$

Example 1.5 - Limits of Piecewise Functions

Do the following limits exist?

(a)

$$\lim_{x\to 0} \frac{x}{|x|} \tag{1.46}$$

(b)

$$\lim_{x\to \pi/2} \frac{|\sin 2x|}{\sin x} \tag{1.47}$$

(a) Since the function changes form either side of the limit, we must evaluate the left and right-sided limits separately. The left sided limit is

$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{(-x)} = -1 \tag{1.48}$$

The right-sided limit is

$$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{x} = 1 \tag{1.49}$$

Since these don't match, the limit does not exist.

(b) Left-sided limit

$$\lim_{x\to \pi/2^-} \frac{|\sin 2x|}{\sin x} = \lim_{x\to \pi/2^-} \frac{\sin 2x}{\sin x} = \frac{0}{1} = 0 \tag{1.50}$$

Right-sided limit

$$\lim_{x\to \pi/2^+} \frac{|\sin 2x|}{\sin x} = \lim_{x\to \pi/2^+} \frac{-\sin 2x}{\sin x} = \frac{0}{1} = 0 \tag{1.51}$$

And so the limit does exist.[†]

[†] In fact, we can go further and say the function is continuous here since $f(\pi/2) = 0$ as well.

2 Continuity

- Know definition of continuity
- Evaluate continuity at points in piecewise functions
- Know different types of discontinuity (removable, infinite, jump)
- Use the IVT to determine existence of roots

Example 2.1 - Evaluating continuity of piecewise funtions

Sketch the following functions and at each discontinuity, state whether f is (left / right) continuous and the type of discontinuity.

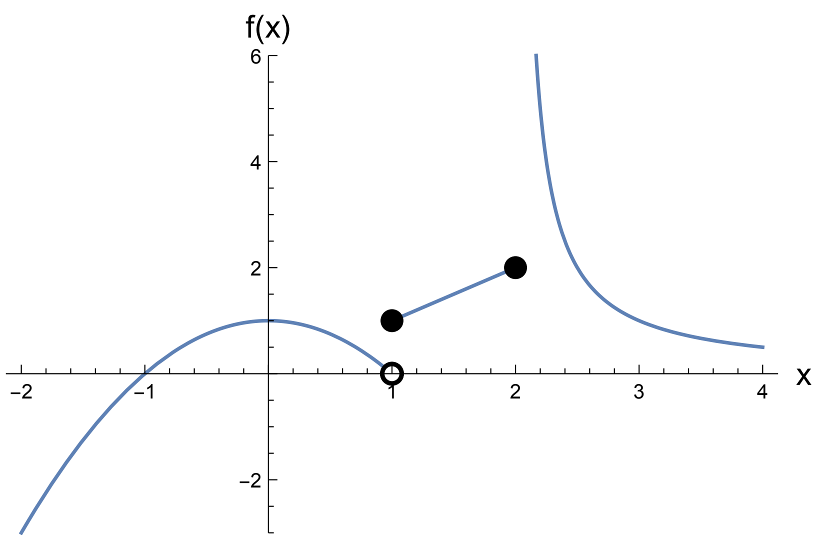
(a)

$$f(x) = \begin{cases} -x^2+1 & x<1 \\ x & 1 \leq x \leq 2 \\ \frac{1}{x-2} & x>2 \end{cases} \tag{2.1}$$

(b)

$$g(x) = \begin{cases} x^2+1 & x<0 \\ 0 & x=0 \\ \cos\left(\frac{x}{4}\right) & x>0 \end{cases} \tag{2.2}$$

(a) Sketch:



- Discontinuities occur at $x = 1$ and $x = 2$.
- Around $x = 1$

$$\lim_{x\to 1^-} f(x) = 0, \qquad \lim_{x\to 1^+} f(x) = 1, \qquad f(1) = 1 \tag{2.3}$$

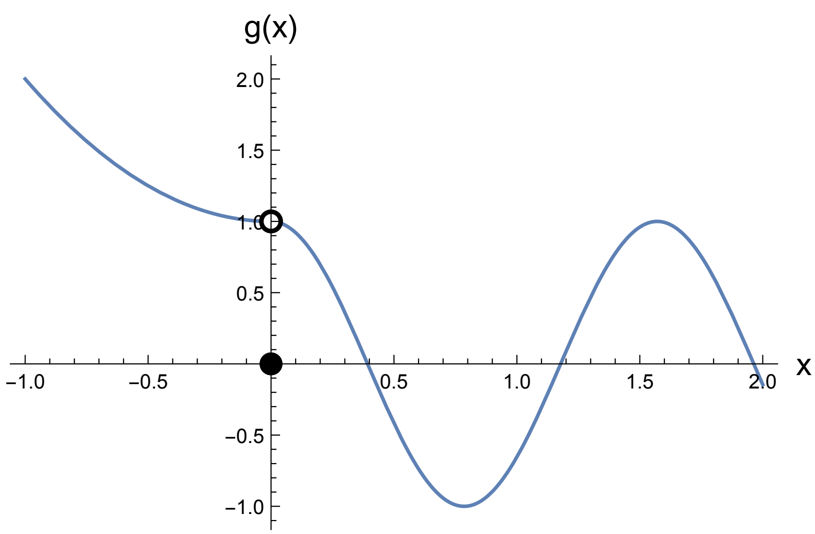
Therefore f is right-continuous at $x = 1$ and there is a jump discontinuity here.

- Around $x = 2$

$$\lim_{x\to 2^-} f(x) = 2, \qquad \lim_{x\to 2^+} f(x) = \infty, \qquad f(2) = 2 \tag{2.4}$$

Therefore f is left-continuous at $x = 2$ and there is an infinite discontinuity here.

(b) Sketch:



- Around $x = 0$

$$\lim_{x\to 0^-} g(x) = 1, \qquad \lim_{x\to 0^+} g(x) = 1, \qquad g(0) = 0 \tag{2.5}$$

Therefore $g(x)$ is neither continuous from the right or the left at $x = 0$. The point $x = 0$ is a removable singularity since the left and right limits are equal.

Example 2.2 - The Intermediate Value Theorem

(a) Show that

$$f(x) = \frac{x^7}{x^5 + 1} \tag{2.6}$$

takes on the value 0.32.

(b) Show that

$$e^x + \ln x = 0 \tag{2.7}$$

has a solution.

(a)

- Recall IVT: *If a function f is continuous on the closed interval $[a, b]$, then for any number M that lies in between $f(a)$ and $f(b)$, there exists a $c \in (a, b)$ such that $f(c) = M$.*
- This function is continuous on the interval $[0, 1]$.
- $f(0) = 0$, $f(1) = 0.5$
- Since 0.32 lies between $f(0)$ and $f(1)$, and f is continuous on this interval, there exists a c such that $f(c) = 0.32$ by the IVT.

(b)

- Let $f(x) = e^x + \ln x$. Note that f is continuous for $x > 0$.
- Pick some values...$f(1) = e > 0$
- $f(1/100) = e^{1/100} - \ln(100) < 0$. (Just pick any value that makes $f < 0$.)
- Now by the IVT there exists a $c \in [1/100, 1]$ such that $f(c) = 0$
- i.e there exists a solution to $e^x + \ln x = 0$.