Examples 5
Differential Calculus

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The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

1 Differential Calculus

- Find derivatives from first principles
- Determine differentiability of a function
- Know when to use, and how to implement implicit differentiation
- Compute derivatives of inverse functions
- Apply logarithmic differentiation when convenient

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Example 1.1 - Derivatives from first principles

Using the definition of a derivative, compute $f'(x)$ for the following functions:

(a) $f(x) = x^2$

(b) $f(x) = \sqrt{x^2+1}$

(c) $f(x) = \frac{1}{\sqrt{x}}$

Recall that the derivative of f at a point x is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{1.1}$$

assuming the limit exists. If you prefer, you may also use

$$f'(x) = \lim_{y\to x} \frac{f(y) - f(x)}{y - x}. \tag{1.2}$$

(a) Using the definition as given in (1.1) we have

$$f(x) = x^2 \tag{1.3}$$
$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \tag{1.4}$$
$$= \lim_{h\to 0} \frac{2xh + h^2}{h} \tag{1.5}$$
$$= \lim_{h\to 0} (2x + h) \tag{1.6}$$
$$= 2x \tag{1.7}$$

as of course we all knew already. We'll use the other method too, just this once...

$$f'(x) = \lim_{y\to x} \frac{y^2 - x^2}{y - x} \tag{1.8}$$
$$= \lim_{y\to x} \frac{(y+x)(y-x)}{y - x} \tag{1.9}$$
$$= \lim_{y\to x} y + x \tag{1.10}$$
$$= 2x. \tag{1.11}$$

In hindsight I would recommend the first definition for these exercises - having a simple denominator makes life easier.

(b) From (1.1) we have

$$f(x) = \sqrt{x^2+1} \tag{1.12}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \tag{1.13}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(x+h)^2+1-(x^2+1)}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \qquad \text{multiply by conjugate} \tag{1.14}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{2xh+h^2}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \tag{1.15}$$

$$= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \tag{1.16}$$

$$= \frac{x}{\sqrt{x^2+1}} \tag{1.17}$$

(c) From (1.1) we have

$$f(x) = \frac{1}{\sqrt{x}} \tag{1.18}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \tag{1.19}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \tag{1.20}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \qquad \text{multiply by conjugate} \tag{1.21}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \tag{1.22}$$

$$= -\frac{1}{2x^{3/2}} \tag{1.23}$$

Example 1.2 - Differentiability of functions

Determine at which points the following functions are differentiable.

(a) $f(x) = |x - 2|$

(b) $f(x) = x + H(x - 1)$

(c) $f(x) = 4x + (1 - 4x + \ln 4x)H(x - 1/4)$, may use $\lim_{h\to 0} \frac{\ln(h+1)}{h} = 1$

Recall that $f(x)$ if differentiable at the point x_0 if the limit

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1.24}$$

exists. I.e if the derivative at the point x_0 is defined.

(a) First note that

$$f(x) = |x - 2| = \begin{cases} 2 - x & x < 2 \\ x - 2 & x \geq 2 \end{cases} \tag{1.25}$$

We can see immediately that x is differentiable on the open intervals $(-\infty, 2)$ and $(2, \infty)$: we can calculate the derivatives on these intervals as -1 and 1 respectively.

At $x = 2$ we must check to see if the limit defining the derivative exists. We have

$$\lim_{h\to 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h\to 0^-} \frac{2 - (2 + h) - 0}{h} = -1. \tag{1.26}$$

However,

$$\lim_{h\to 0^+} \frac{f(2 + h) - f(2)}{h} = \lim_{h\to 0^+} \frac{2 + h - 2 - 0}{h} = 1. \tag{1.27}$$

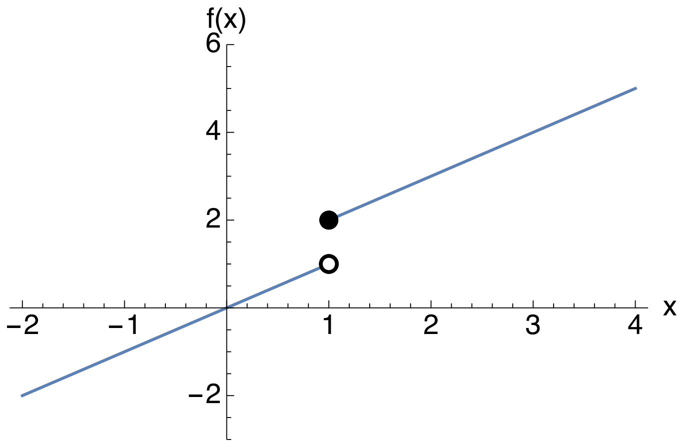
Since the this limit is not defined at $x = 2$, f is not differentiable there. We conclude **f is only differentiable on the interval** $(-\infty, \mathbf{2}) \cup (\mathbf{2}, \infty)$

Not all continuous functions are differentiable!

(b) Write

$$f(x) = x + H(x-1) = \begin{cases} x & x < 1 \\ x+1 & x \geq 1 \end{cases} \tag{1.28}$$

Sketch:



This function is differentiable for $x \in (-\infty, 1) \cup (1, \infty)$ and has derivative 1. It has the same derivative either side of $x = 1$...does that mean it's differentiable at $x = 1$? Around this point we have

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{1 + h + 1 - 2}{h} = 1 \tag{1.29}$$

and

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{1 + h - 2}{h} = \lim_{h \to 0^-} 1 - \frac{1}{h} = \infty \tag{1.30}$$

and so the limit is not defined. In fact, one can prove

$$\mathbf{f \text{ differentiable at } x_0 \quad \Rightarrow \quad f \text{ continuous at } x_0} \tag{1.31}$$

and thus its contrapositive

$$\mathbf{f \text{ discontinuous at } x_0 \quad \Rightarrow \quad f \text{ NOT differentiable at } x_0}. \tag{1.32}$$

(c) Write

$$f(x) = \begin{cases} 4x & x < 1/4 \\ 1 + \ln(4x) & x \geq 1/4 \end{cases} \tag{1.33}$$

On $(-\infty, 1/4)$ we have $f'(x) = 4$

On $(1/4, \infty)$ we have $f'(x) = \frac{1}{x}$

We see that f is continuous at $x = \frac{1}{4}$ so it *may* be differentiable.

Check limits:

$$\lim_{h \to 0^-} \frac{f(1/4 + h) - f(1/4)}{h} = \lim_{h \to 0^-} \frac{1 + 4h - 1}{h} = 4, \tag{1.34}$$

$$\lim_{h \to 0^+} \frac{f(1/4 + h) - f(1/4)}{h} = \lim_{h \to 0^+} \frac{1 + \ln(1 + 4h) - 1}{h} \tag{1.35}$$

$$= \lim_{h \to 0^+} \frac{\ln(1 + 4h)}{h} \tag{1.36}$$

$$= 4 \lim_{u \to 0^+} \frac{\ln(1 + u)}{u} \qquad \text{setting } u = 4h \tag{1.37}$$

$$= 4 \tag{1.38}$$

This function is therefore differentiable everywhere.

Example 1.3 - Implicit Differentiation

(a) Find y' for the following relations. Answers may be left in terms of x and y.

(i) $y^2 + x^2 = 1$

(ii) $\sin(y^3 - xy) = \cos(x^2 - 3)$

(b) Find the tangent line to the curve

$$e^{xy} - 2 = 0 \tag{1.39}$$

at the point $(x, y) = (1, \ln 2)$ *using implicit differentiation.* Check your answer by rearranging for y.

(a)

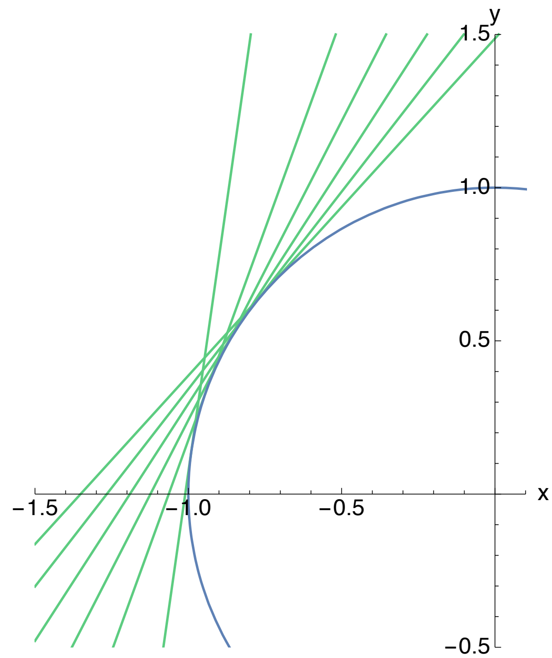
(i) This is of course the unit circle. Differentiating w.r.t. x we have

$$x^2 + y^2 = 1 \tag{1.40}$$

$$\Rightarrow \quad 2x + 2yy' = 0 \tag{1.41}$$

$$\Rightarrow \quad y' = -\frac{x}{y} \tag{1.42}$$

The derivative is not defined at $x = \pm 1$ $(y = 0)$ as expected.



(ii) Messy expressions involving x and y like this should be differentiated implicitly:

$$\sin(y^3 - xy) = \cos(x^2 - 3) \tag{1.43}$$

$$\Rightarrow \quad \cos(y^3 - xy)\left(3y^2y' - (y + xy')\right) = -\sin(x^2 - 3)2x \qquad \text{using chain and product rule} \tag{1.44}$$

$$\Rightarrow \quad y'(3y^2 - x) - y = -\frac{2x\sin(x^2 - 3)}{\cos(y^3 - xy)} \tag{1.45}$$

$$\Rightarrow \quad y' = \frac{1}{3y^2 - x}\left(y - \frac{2x\sin(x^2 - 3)}{\cos(y^3 - xy)}\right) \tag{1.46}$$

(b) Differentiating implicitly we have

$$e^{xy} - 2 = 0 \tag{1.47}$$

$$\Rightarrow \quad e^{xy}(y + xy') = 0 \tag{1.48}$$

$$\Rightarrow y' = -\frac{y}{x} \tag{1.49}$$

At $(x_0, y_0) = (1, \ln 2)$ we have $y' = -\ln 2$. The straight line going through the point (x_0, y_0) with gradient m is

$$y - y_0 = m(x - x_0) \tag{1.50}$$

And so the equation of the tangent line is

$$y - \ln 2 = -\ln 2(x - 1) \tag{1.51}$$

$$\Rightarrow \quad y = \ln 2(-x + 2) \tag{1.52}$$

For this simpler case, we could have rearranged (1.48) to get $y = (\ln 2)/x$ and differentiated normally to get the same result.

Example 1.4 - Derivatives of inverse functions

Find the derivative of the following functions

(a) $f(x) = \cos^{-1}(x)$

(b) $f(x) = \sinh^{-1}(x)$

In part (b) you may use results

$$\cosh^2(x) - \sinh^2(x) = 1, \quad \frac{d}{dx}\sinh(x) = \cosh(x), \quad \cosh(x) > 0 \quad \text{for} \quad x \in \mathbb{R}. \tag{1.53}$$

(a) Set $y = \cos^{-1}(x)$, so $y \in [0, \pi]$. Then

$$x = \cos y \tag{1.54}$$

$$\Rightarrow \quad \frac{dx}{dy} = -\sin y \tag{1.55}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\sin y} \qquad \text{using} \quad \frac{dy}{dx} = \frac{1}{dx/dy} \tag{1.56}$$

We would like the derivative in terms of x since y is something we introduced. Since $y \in [0, \pi]$, $\sin y > 0$ and so we may use

$$\sin y = \sqrt{1 - \cos^2 y} \tag{1.57}$$

$$= \sqrt{1 - x^2}. \tag{1.58}$$

Then

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}} \tag{1.59}$$

(b) Set $y = \sinh^{-1}(x)$. Then

$$x = \sinh(y) \tag{1.60}$$

$$\Rightarrow \quad \frac{dx}{dy} = \cosh(y) = \sqrt{1 + \sinh^2(y)} = \sqrt{1 + x^2} \tag{1.61}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}} \tag{1.62}$$

where the positive root was taken in (1.61) since $\cosh(y) > 0$.

Example 1.5 - Logarithmic differentiation

For derivatives of functions involving the product of many terms raised to powers, logarithmic differentiation provides us with a shortcut. Find the derivative of the following functions:

(a) $f(x) = \frac{(x+1)^{3/2}\sin^2(x)}{x-1}$

(b) $f(x) = x^x$

(c) $f(x) = x^{x^x}$ (bonus q if time)

(a) Take the natural logarithm of both sides to break the product up in to a sum...

$$f(x) = \frac{(x+1)^{3/2}\sin^2(x)}{x-1} \tag{1.63}$$

$$\Rightarrow \quad \ln f(x) = \frac{3}{2}\ln(x+1) + 2\ln(\sin x) - \ln(x-1) \tag{1.64}$$

Differentiating both sides with respect to x gives

$$\frac{f'(x)}{f(x)} = \frac{3}{2(x+1)} + \frac{2\cos x}{\sin x} - \frac{1}{x-1} \tag{1.65}$$

$$\Rightarrow \quad f'(x) = \frac{(x+1)^{3/2}\sin^2(x)}{x-1}\left(\frac{3}{2(x+1)} + \frac{2\cos x}{\sin x} - \frac{1}{x-1}\right) \tag{1.66}$$

(b) By the same procedure

$$f(x) = x^x \tag{1.67}$$

$$\Rightarrow \quad \ln f(x) = x\ln x \tag{1.68}$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = 1 + \ln x \tag{1.69}$$

$$\Rightarrow \quad f'(x) = x^x(1 + \ln x) \tag{1.70}$$

(c) And now....

$$f(x) = x^{x^x} \tag{1.71}$$

$$\Rightarrow \quad \ln f(x) = x^x \ln x \tag{1.72}$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = (x^x)'\ln x + x^x(\ln x)' = x^x(1+\ln x)\ln x + x^{x-1} \tag{1.73}$$

$$\Rightarrow \quad f'(x) = x^{x^x}\left(x^x(1+\ln x)\ln x + x^{x-1}\right) \tag{1.74}$$