Examples 6
Differentials, L'Hopital's Rule, and Curve Sketching

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The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

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1 Differentials

- Use differentials to approximate values
- Manipulate differentials to give percentage error estimates

Example 1.1 - Estimation

Use differentials to find estimates for the following values

(a) $\sqrt{51}$

(b) $\sin\left(\frac{\pi}{4}+0.1\right)$

(a) **Using differentials**

- We can easily evaluate $\sqrt{49}$ so we choose $x_0 = 49$ as our base point for the function $f(x) = \sqrt{x}$.
- Calculate the differential of f:

$$df = f'(x)dx \tag{1.1}$$

$$= \frac{1}{2\sqrt{x}}dx \tag{1.2}$$

- Now set $dx = 2$ and $x = 49$

$$df = \frac{1}{2\sqrt{49}} * 2 = \frac{1}{7} \tag{1.3}$$

and so

$$\Delta f \approx \frac{1}{7} \tag{1.4}$$

- This is the change in f from the point $x_0 = 49$ so we have

$$f(52) = f(49) + \Delta f \tag{1.5}$$

$$\approx 7 + \frac{1}{7} \tag{1.6}$$

$$= \frac{50}{7} \tag{1.7}$$

Using a taylor expansion (same methodology but more concise)

- The first order Taylor expansion gives

$$f(x_0 + \Delta x) \approx f(x_0) + \Delta x f'(x_0) \tag{1.8}$$

$$\Rightarrow f(51) \approx f(49) + 2 * f'(49) \tag{1.9}$$

$$= 7 + 2 * \frac{1}{2 * 7} = \frac{50}{7} \tag{1.10}$$

(b) Using the Taylor expansion

$$\sin\left(\frac{\pi}{4}+0.1\right) \approx \sin\left(\frac{\pi}{4}\right)+0.1\cos\left(\frac{\pi}{4}\right) \tag{1.11}$$

$$= \frac{1}{\sqrt{2}}+0.1\frac{1}{\sqrt{2}} \tag{1.12}$$

$$= \frac{11}{10\sqrt{2}} \tag{1.13}$$

A quick note: Where did that Taylor expansion (1.8) come from? Rearrange it to give

$$f'(x_0) \approx \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

Approximation becomes equality as $\Delta x \to 0$ and of course this is the definition of the derivative. More on Taylor Series in Math 119.

Example 1.2 - Percentage Error

A soccer ball manufacturer wishes to make a ball of volume V, allowing a maximum of a 3% percentage error. Estimate the maximum percentage error in the diameter of the ball required to achieve this.

The volume of a sphere with diameter l is given by $V = \frac{1}{6}\pi l^3$.

- Max % error in V is 3% so

$$\left|\frac{dV}{V}\right| < 0.03. \tag{1.14}$$

- The differential of V is

$$dV = \frac{1}{2}\pi l^2\, dl. \tag{1.15}$$

- Then

$$\frac{dV}{V} = \frac{\frac{1}{2}\pi l^2\, dl}{\frac{1}{6}\pi l^3} = 3\frac{dl}{l}. \tag{1.16}$$

- And so

$$\left|\frac{dl}{l}\right| = \frac{1}{3}\left|\frac{dV}{V}\right| < 0.01, \tag{1.17}$$

which means we require a percentage error in diameter less than 1%.

(Remember this is approximate - differentials provide approximations when dealing with finite changes)

2 L'Hopital's Rule

- Evaluate limits with indeterminate forms such as "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$0.\infty$", "1^∞"

Example 2.1 - Forms "$\frac{0}{0}$" "$\frac{\infty}{\infty}$"

Evaluate the following limits using L'Hopital's Rule:

(a)

$$\lim_{x\to 0}\frac{\sin x}{x} \tag{2.1}$$

(b)

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} \tag{2.2}$$

(c)

$$\lim_{x\to \infty}\frac{\ln x}{x} \tag{2.3}$$

(a) This has the indeterminate form "0/0".

$$\lim_{x\to 0}\frac{\sin x}{x} \stackrel{H}{=} \lim_{x\to 0}\frac{\cos x}{1} = 1 \tag{2.4}$$

(b) L'Hopital's rule can be applied multiple times if we continue to get indeterminate forms:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} \stackrel{H}{=} \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} \quad \text{form "0/0"} \tag{2.5}$$

$$\stackrel{H}{=} \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} \quad \text{using } (\sec x)' = \sec x\tan x \tag{2.6}$$

Now before we dive into another l'Hopital, it saves time breaking the limits up into products:

$$= \frac{1}{3}\left(\lim_{x\to 0}\sec^2 x\right)\left(\lim_{x\to 0}\frac{\tan x}{x}\right) \quad \text{second bracket "0/0"} \tag{2.7}$$

$$\stackrel{H}{=} \frac{1}{3}\lim_{x\to 0}\frac{\sec^2 x}{1} \tag{2.8}$$

$$= \frac{1}{3} \tag{2.9}$$

(c) This has the indeterminate form "∞/∞":

$$\lim_{x\to\infty}\frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0 \tag{2.10}$$

Loosely speaking, x goes to infinity faster than $\ln x$.

Example 2.2 - Forms "$0.\infty$", "1^∞"

Using L'Hopital's Rule, evaluate the following limits:

(a)

$$\lim_{x\to 0^+} xe^{1/x} \tag{2.11}$$

(b)

$$\lim_{x\to 0}(1+x)^{1/x} \tag{2.12}$$

(a) - This has the indeterminate form "$0*\infty$". Note that if the limit was from below ($x\to 0^-$) we would have the form "$0*0$" which you can automatically say is 0.

- We can put these types of limit into the form "0/0" or "∞/∞", whichever makes life easier:

$$\lim_{x\to 0^+} xe^{1/x} = \lim_{x\to 0^+}\frac{e^{1/x}}{x^{-1}} \quad \text{this is now in the form "}\infty/\infty\text{"} \tag{2.13}$$

$$= \lim_{x\to 0^+}\frac{-\frac{1}{x^2}e^{1/x}}{-\frac{1}{x^2}} \tag{2.14}$$

$$\overset{H}{=} \lim_{x\to 0^+} e^{1/x} \tag{2.15}$$

$$= \infty \tag{2.16}$$

- If you like substitutions, we could have simplified the exponent using $u=\frac{1}{x}$ then

$$\lim_{x\to 0^+} xe^{1/x} = \lim_{u\to\infty}\frac{e^u}{u} \overset{H}{=} \lim_{u\to\infty}\frac{e^u}{1} = \infty. \tag{2.17}$$

(b) - This has the indeterminate form "1^∞", should this be 1, ∞, neither??

- We proceed by taking logarithms : let $y = (1+x)^{\frac{1}{x}}$. Then

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{1}{x} \ln(1+x) \quad \overset{H}{=} \quad \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1 \tag{2.18}$$

So the limit of its logarithm wasn't too bad. The trick now is, since $\ln y$ is continuous everywhere,

$$\ln\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \ln y = 1. \tag{2.19}$$

And so

$$\lim_{x\to 0} (1+x)^{1/x} = e. \tag{2.20}$$

...nice

3 Curve sketching and optimisation techniques

- Identify critical points of functions and their properties (local max/min/neither)
- Locate absolute extrema using the closed interval method
- Sketch curves with the help of derivative tests

Example 3.1 - Finding Absolute Extrema - (Closed Interval Method)

Find the absolute maximum and absolute minimum values of the following functions on their respective domains:

(a)

$$f(x) = \frac{\ln x}{x}, \qquad x \in [1, e^2] \tag{3.1}$$

(b)

$$f(x) = |\cos x|, \qquad x \in [0, \tfrac{5\pi}{4}] \tag{3.2}$$

(a) - ***Locate the critical points:***

$$f'(x) = \frac{1 - \ln x}{x^2} \tag{3.3}$$

which is zero at $x = e$ and undefined at $x = 0$. $f(x)$ is also undefined at $x = 0$ so only $x = e$ is a critical point.

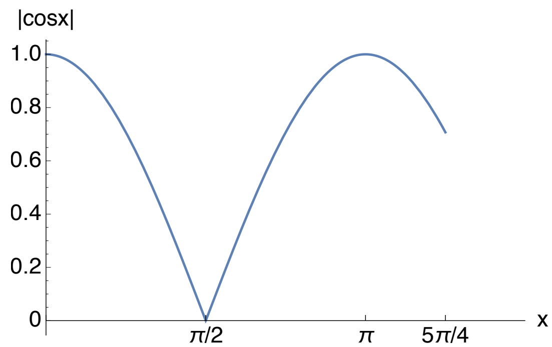
- ***Compare values for f at critical points and edge points:***

$$f(1) = 0, \qquad f(e) = e^{-1}, \qquad f(e^2) = 2e^{-2} \tag{3.4}$$

And so the absolute min is $f = 0$ and the absolute max is $f = e^{-1}$.

To convince yourself that $2e^{-2} < e^{-1}$, note that $e > 2 \Rightarrow 2e^{-1} < 1 \Rightarrow 2e^{-2} < e^{-1}$.

(b) ***If a sketch is quick - draw it for intuition***



Locate the critical points:

Before differentiating functions with absolute value signs, write them in piecewise form!

$$f(x) = \begin{cases} \cos x & x \in [0, \frac{\pi}{2}] \\ -\cos x & x \in [\pi/2, \frac{5\pi}{4}] \end{cases} \tag{3.5}$$

Then

$$f'(x) = \begin{cases} -\sin x & x \in (0, \frac{\pi}{2}) \\ \sin x & x \in (\frac{\pi}{2}, \frac{5\pi}{4}) \end{cases} \tag{3.6}$$

We have $f'(x) = 0$ at $x = \pi$ and f' is probably not defined at $x = \pi/2$.

Check:

$$\lim_{h \to 0^-} \frac{f(\frac{\pi}{2} + h) - f(\frac{\pi}{2})}{h} = \lim_{h \to 0^-} \frac{\cos(\frac{\pi}{2} + h) - \cos(\frac{\pi}{2})}{h} \tag{3.7}$$

$$= \lim_{h \to 0^-} \frac{\cos(\frac{\pi}{2}) \cos h - \sin(\frac{\pi}{2}) \sin h}{h} \tag{3.8}$$

$$= -\lim_{h \to 0^-} \frac{\sin h}{h} = -1 \tag{3.9}$$

Similarly we can show the right-sided limit is $+1$.

So the critical points are $x = \frac{\pi}{2}$ and $x = \pi$.

Compare values of f at critical points and end points:

$$f(0) = 1, \qquad f(\pi/2) = 0, \qquad f(\pi) = 1, \qquad f(5\pi/4) = 1/\sqrt{2} \tag{3.10}$$

And so $f_{min} = 0$, $f_{max} = 1$.

Example 3.2 - Curve sketching

Sketch the following functions, illustrating behaviour at critical points and points of inflection.

(a)
$$f(x) = -2x^3 + 9x^2 - 12x + 6 \tag{3.11}$$

(b)
$$f(x) = x^2 \ln x, \qquad x \in (0, \infty) \tag{3.12}$$

(a) - ***Find the critical points:***

$$f'(x) = -6x^2 + 18x - 12 \tag{3.13}$$
$$= -6(x-1)(x-2) \tag{3.14}$$

which is zero at $x = 1$ and $x = 2$. They are the critical points.

- ***Evaluate the behaviour at the critical points:***

Using the First Derivative Test
Evaluate the sign of $f'(x)$ either side of the critical points:

$$f'(x) < 0 \quad \text{for } x < 1 \tag{3.15}$$
$$f'(x) > 0 \quad \text{for } 1 < x < 2 \tag{3.16}$$
$$f'(x) < 0 \quad \text{for } x > 2 \tag{3.17}$$

Therefore $x = 1$ is a local minimum, and $x = 2$ is a local maximum.

Or using the Second Derivative Test
Evaluate the sign of $f''(x)$ *at* the critical points:

$$f''(x) = -12x + 18 \tag{3.18}$$
$$f''(1) = 6 > 0 \tag{3.19}$$
$$f''(2) = -6 < 0 \tag{3.20}$$

in agreement with the first test.

- ***Inflection points:***
  There is an inflection point where

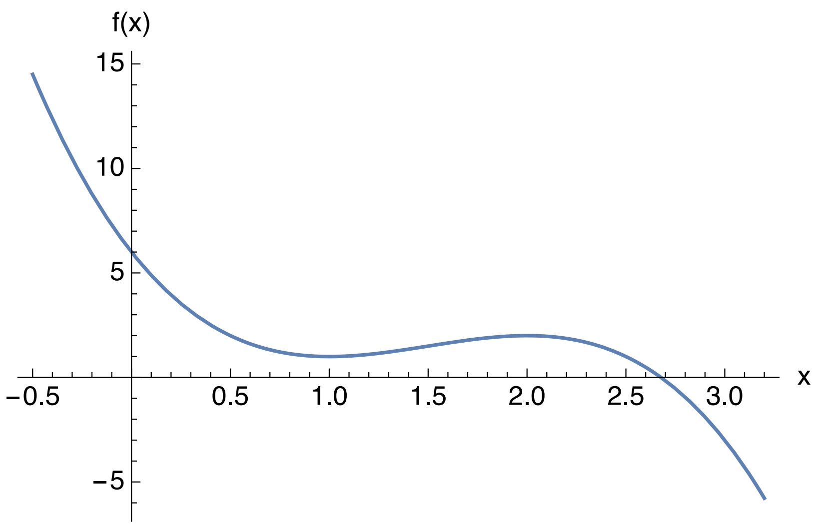
$$f''(x) = 0 \tag{3.21}$$

$$\Rightarrow \quad -12x + 18 = 0 \tag{3.22}$$

$$\Rightarrow \quad x = 3/2 \tag{3.23}$$

- ***Make a sketch***

  Also consider behaviour as $x \to \pm\infty$ to help.



(b) $f(x) = x^2 \ln x, \quad x \in (0, \infty)$

Investigate behaviour towards interval bounds:

$$\lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-2}} \overset{H}{=} \lim_{x \to 0^+} \frac{x^{-1}}{-2x^{-3}} = \lim_{x \to 0^+} \frac{x^2}{-2} = 0 \tag{3.24}$$

Note that for very small values of x, $f(x)$ is negative, so it tends to zero from below.

$$\lim_{x \to \infty} x^2 \ln x = \infty \tag{3.25}$$

Find critical points:

$$f'(x) = 2x \ln x + x^2(1/x) = x(1 + 2\ln x) \tag{3.26}$$

which is zero at $x = 0$ (discard since not in the domain of f), and at $x = e^{-1/2}$.

So there is a single critical point at $x = e^{-1/2}$.

Investigate nature of critical points:

Using the First Derivative Test

Note that $f(x) < 0$ for $x \in (0, e^{-1/2})$ and $f'(x) > 0$ for $x \in (e^{-1/2}, \infty)$. Therefore $x = e^{-1/2}$ is a local min.

Or Using the Second Derivative Test

The second derivative is

$$f''(x) = (1 + 2\ln x) + x(2/x) = 3 + 2\ln x \tag{3.27}$$

and so

$$f''(e^{-1/2}) = 3 + 2\ln(e^{-1/2}) = 3 + 2(-1/2) = 2 \tag{3.28}$$

which is > 0 as expected, representing a local min.

Inflection Points:

Inflection points occur at $f''(x) = 0$ i.e

$$3 + 2\ln x = 0 \tag{3.29}$$

$$\Rightarrow \quad x = e^{-3/2} \tag{3.30}$$

Sketch:

Note that f(1) = 0.