Examples 7
Riemann Integrals, The Fundamental Theorem of Calculus and Integration Techniques

November 14, 2016

The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

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1 Riemann Integrals

- Compute integrals from first principles using the definition of the Riemann integral

Example 1

Calculate the following integrals as limits of Riemann sums:

(a)
$$\int_0^2 3\,dx \tag{1.1}$$

(b)
$$\int_1^2 x^3\,dx \tag{1.2}$$

Recall the definition of the definite (Riemann) integral for a continuous function f on $[a,b]$:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \tag{1.3}$$

where $\Delta x = (b-a)/n$, $x_i^* \in [x_{i-1}, x_i]$ and $x_i = a + i\Delta x$.

We may use the following identities:

$$\sum_{k=1}^{n} 1 = n \qquad \sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \qquad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{1.4}$$

(a) A diagram shows straight away that this area is 6, however let's check that the definition agrees.

- ***Segment width***
$$\Delta x = \frac{b-a}{n} = \frac{2}{n} \tag{1.5}$$

- ***Segment evaluation point*** $\mathbf{x}^*$

  Choose $x_i^* = x_i$ (right side of segment) so
$$x_i^* = a + i\Delta x = \frac{2i}{n}. \tag{1.6}$$

- ***The Riemann Sum***

$$R_n = \sum_{i=1}^{n} 3.\frac{2}{n} = \frac{6}{n}\sum_{i=1}^{n} 1 = 6 \tag{1.7}$$

- ***Take the limit***

$$\int_0^2 3dx = \lim_{n\to\infty} R_n = 6 \tag{1.8}$$

Note that in this case the Riemann sum does not depend on n (the number of segments we divide the interval up in to). This is because the area is already a rectangle!

(b) A bit harder:

- ***Segment width***

$$\Delta x = \frac{b-a}{n} = \frac{1}{n} \tag{1.9}$$

- ***Segment evaluation point* $\mathbf{x}^*$**

$$x_i^* = a + i\Delta x = 1 + \frac{i}{n}. \tag{1.10}$$

- ***The Riemann Sum***

$$R_n = \sum_{i=1}^{n} (x_i^*)^3 \Delta x \tag{1.11}$$

$$= \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^3 \left(\frac{1}{n}\right) \tag{1.12}$$

$$= \sum_{i=1}^{n} \left(1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3}\right)\left(\frac{1}{n}\right) \tag{1.13}$$

$$= \sum_{i=1}^{n} 1 + \frac{3}{n}\sum_{i=1}^{n} i + \frac{3}{n^2}\sum_{i=1}^{n} i^2 + \frac{1}{n^3}\sum_{i=1}^{n} i^3 \left(\frac{1}{n}\right) \tag{1.14}$$

$$= 1 + \frac{3}{n^2}\left(\frac{1}{2}n(n+1)\right) + \frac{3}{n^3}\left(\frac{1}{6}n(n+1)(2n+1)\right) + \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 \tag{1.15}$$

$$\tag{1.16}$$

- ***Take the limit***

$$\int_1^2 x^3 dx = \lim_{n\to\infty} R_n \tag{1.17}$$

$$= 1 + \frac{3}{2} + 1 + \frac{1}{4} = \frac{15}{4} \tag{1.18}$$

2 The Fundamental Theorem of Calculus

- Take derivatives of integrals using the FTC Part I
- Compute definite integrals using the FTC Part II

Example 2.1- derivatives of integrals

Differentiate the following functions

(a)

$$f(x) = \int_2^{x^2} e^{-t^2} dt \tag{2.1}$$

(b)

$$f(x) = \int_{\cos x}^{\sin x} \sqrt{1+t^2} dt \tag{2.2}$$

Recall that the FTC Part I gives us the differentiation rule

$$\frac{d}{dx} \int_a^x g(t) dt = g(x) \tag{2.3}$$

(a) Let $u = x^2$ and then use the chain rule:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \tag{2.4}$$

$$= \frac{d}{du}\left(\int_2^u e^{-t^2} dt\right) .2x \tag{2.5}$$

$$= e^{-u^2}.2x \qquad \text{from the FTC} \tag{2.6}$$

$$= 2xe^{-x^4} \tag{2.7}$$

(b) We break the integral up into forms that permit the FTC. Useful identities are

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt \tag{2.8}$$

$$\int_a^b f(t)dt = -\int_b^a f(t)dt. \tag{2.9}$$

We may therefore write

$$
\begin{aligned}
f(x) &= \int_{\cos x}^{\sin x} \sqrt{1+t^2}dt && (2.10)\\
&= \int_{\cos x}^{0} \sqrt{1+t^2}dt + \int_{0}^{\sin x} \sqrt{1+t^2}dt \qquad \text{we could have used any constant instead of 0} && (2.11)\\
&= -\int_{0}^{\cos x} \sqrt{1+t^2}dt + \int_{0}^{\sin x} \sqrt{1+t^2}dt && (2.12)\\
&= -\sqrt{1+\cos^2 x}(-\sin x) + \sqrt{1+\sin^2 x}\cos x \qquad \text{using methods from (a)} && (2.13)\\
&= \sin x\sqrt{1+\cos^2 x} + \cos x\sqrt{1+\sin^2 x} && (2.14)
\end{aligned}
$$

Example 2.2 - definite integrals

Evaluate the following:

(a)

$$\int_1^2 x^3 dx \tag{2.15}$$

(b)

$$\int_{\frac{\pi}{2}}^{x} \cos t \, dt \tag{2.16}$$

Recall that the FTC Part II tells us that if $F(x)$ is the anti-derivative of $f(x)$, then

$$\int_a^b f(t)dt = F(a) - F(b) \tag{2.17}$$

(a) The antiderivative of x^3 is $\frac{1}{4}x^4$ so FTC II tells us

$$\int_1^2 x^3 dx = \frac{1}{4}x^4 \Big|_1^2 = \frac{15}{4} \tag{2.18}$$

The FTC II saves us a lot of time cf. Example 1.(a).

(b) The antiderivative of $\cos t$ is $\sin t$ and so

$$\int_{\frac{\pi}{2}}^{x} \cos t dt = \sin t \Big|_{\frac{\pi}{2}}^{x} = \sin x - 1 \tag{2.19}$$

Note that this is consistent with FTC I if we were to differentiate both sides.

3 Integration Techniques

- Recognise even and odd integrands to save time
- Integrate piecewise-defined functions
- Use a change of variables to simplify the integral
- Employ integration by parts when appropriate

Example 3.1 - odd / even / Heaviside integrands

Evaluate the following integrals

(a)

$$\int_{-1}^{1} (x^4 + x^2) dx \tag{3.1}$$

(b)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x e^{-x^2} dx \tag{3.2}$$

(c)

$$\int_{0}^{3} \left[x^2 + H(x-2)(4 - x^2) \right] dx \tag{3.3}$$

Recall that for any even function f,

$$\int_{-a}^{a} f(x) dx = 2 \int_{0}^{a} f(x) dx \tag{3.4}$$

and for an odd function g,

$$\int_{-a}^{a} g(x) dx = 0 \tag{3.5}$$

for any value of a. You may wish to show this algebraically or via a sketch.

(a) The integrand is even so we can save a bit of time using the above result:

$$\int_{-1}^{1} (x^4 + x^2)dx = 2\int_0^1 (x^4 + x^2)dx \tag{3.6}$$
$$= 2\left(\frac{1}{5}x^5 + \frac{1}{3}x^3\Big|_0^1\right) \tag{3.7}$$
$$= 2\left(\frac{1}{5} + \frac{1}{3}\right) \tag{3.8}$$
$$= \frac{16}{15} \tag{3.9}$$

(b) This integral has an odd integrand. Since the limits are symmetrical about zero, the integral must be zero.

(c) For integrals involving piecewise-defined integrands, separate the integral up into the corresponding pieces:

$$\int_0^3 \left[x^2 + H(x-2)(4-x^2)\right] dx = \int_0^2 x^2 dx + \int_2^3 4\, dx \tag{3.10}$$
$$= \frac{1}{3}x^3\Big|_0^2 + 4x\Big|_2^3 \tag{3.11}$$
$$= \frac{8}{3} + 4 = \frac{20}{3} \tag{3.12}$$

Example 3.2 - The Method of Substitution

Evaluate the following integrals

(a)

$$\int \left(1 - \frac{1}{x}\right) \cos(x - \ln x) dx \tag{3.13}$$

(b)

$$\int_0^{\frac{\pi}{2}} e^{\sin\theta} \cos\theta \, d\theta \tag{3.14}$$

(c)

$$\int_0^{\sqrt{3}} x^3 \sqrt{1 + x^2} dx \tag{3.15}$$

Look for a simplifying substitution, ideally one whose derivative is contained in the integrand.

(a) Let $u = x - \ln x$. Then $du = (1 - 1/x)\, dx$ and so

$$\int \left(1 - \frac{1}{x}\right) \cos(x - \ln x)\, dx = \int \cos u \, du \tag{3.16}$$

$$= \sin u + C \tag{3.17}$$

$$= \sin(x - \ln x) + C \tag{3.18}$$

(b) Let $u = \sin\theta$. Then $du = \cos\theta \, d\theta$. Don't forget to convert the limits!

$$\theta = 0 \Rightarrow u = 0 \tag{3.19}$$

$$\theta = \frac{\pi}{2} \Rightarrow u = 1 \tag{3.20}$$

The integral becomes

$$\int_0^{\frac{\pi}{2}} e^{\sin\theta} \cos\theta \, d\theta = \int_0^1 e^u du \tag{3.21}$$

$$= e^u \Big|_0^1 \tag{3.22}$$

$$= e - 1 \tag{3.23}$$

(c) The substitution is a bit less obvious here...simplify the square root with $u = 1 + x^2$. The integral becomes

$$\int_0^{\sqrt{3}} x^3\sqrt{1+x^2}dx = \int_0^{\sqrt{3}} x^2\sqrt{1+x^2}\, x\, dx \tag{3.24}$$

$$= \int_1^4 (u-1)u^{\frac{1}{2}}\frac{du}{2} \tag{3.25}$$

$$= \frac{1}{2}\int_1^4 (u^{\frac{3}{2}} - u^{\frac{1}{2}})du \tag{3.26}$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^4 \tag{3.27}$$

$$= \frac{1}{2}\left(\frac{2}{5}.32 - \frac{2}{3}.8 - \frac{2}{5} + \frac{2}{3}\right) \tag{3.28}$$

$$= \frac{58}{15} \tag{3.29}$$

Example 3.3 - Integration by Parts

Evaluate the following integrals

(a)
$$\int x \cos x dx \tag{3.30}$$

(b)
$$\int x \ln x dx \tag{3.31}$$

(c)
$$\int_1^e \ln x \, dx \tag{3.32}$$

Recall the relation we use for integration by parts:

$$\int u dv = uv - \int v du \tag{3.33}$$

Hints:

- Make sure dv is something we know how to integrate
- Choose a u that simplifies upon differentiation.

(a) Let

$$u = x \qquad \text{simplifies upon differentiation} \tag{3.34}$$
$$dv = \cos x \, dx \qquad \text{we know how to integrate} \tag{3.35}$$

Then

$$du = dx \tag{3.36}$$
$$v = \sin x \tag{3.37}$$

Using (3.33), we have

$$\int x \cos x dx = x \sin x - \int \sin x dx \tag{3.38}$$
$$= x \sin x + \cos x + C \tag{3.39}$$

(b) $\int x \ln x dx$

Now $\ln x$ isn't easy to integrate so we will set

$$u = \ln x, \qquad dv = x\,dx \tag{3.40}$$

Then

$$du = \frac{1}{x}dx, \qquad v = \frac{1}{2}x^2. \tag{3.41}$$

And so

$$\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2.\frac{1}{x}dx \tag{3.42}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x dx \tag{3.43}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C \tag{3.44}$$

(c) $\int_1^e \ln x\,dx$

IBP can also be useful in cases that don't involve the explicit product of two functions. Let

$$u = \ln x, \qquad dv = dx. \tag{3.45}$$

Then

$$du = \frac{1}{x}dx, \qquad v = x. \tag{3.46}$$

There are boundaries to this integral, so we just carry those through into the formula:

$$\int_1^e \ln x\,dx = x \ln x\Big|_0^e - \int_0^e \,dx \tag{3.47}$$

$$= e - (e - 1) \tag{3.48}$$

$$= 1 \tag{3.49}$$