Examples 8
Further Integration Techniques and Applications

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The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

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1 Further Integration Techniques

- Practise using trig / hyperbolic substitutions where appropriate
- Integrate rational functions using their partial fraction decomposition

Example 1.1 - Direct Trig Substitutions

Calculate the following integrals using an appropriate trig substition

(a)

$$\int \sqrt{4-x^2}\,dx \tag{1.1}$$

(b)

$$\int_0^1 \frac{w}{(w^2+1)^{\frac{3}{2}}}dw \tag{1.2}$$

Notes :

When you see the following forms in the integrand, consider the corresponding trig substitution.

Form in integrand	Trig substitution
$a^2 - x^2$	$x = a\sin\theta$
$a^2 + x^2$	$x = a\tan\theta$
$x^2 - a^2$	$x = a\sec\theta$

(a) $\int \sqrt{4-x^2}\,dx$

Set

$$x = 2\sin\theta \quad \Rightarrow \quad dx = 2\cos\theta\,d\theta \tag{1.3}$$

Note that the integrand is valid only for $x \in [-2, 2]$ so we may then assume $\theta \in [-\pi/2, \pi/2]$.

Then

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta \tag{1.4}$$
$$= 4\int \cos^2\theta\,d\theta \qquad \text{since } \cos\theta > 0 \text{ in this range of } \theta \tag{1.5}$$
$$= 2\int (\cos 2\theta + 1)\,d\theta \qquad \text{using } \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1) \tag{1.6}$$
$$= \sin 2\theta + 2\theta + C \tag{1.7}$$
$$\tag{1.8}$$

We should leave our answer in terms of the original variable, x. Using trig identities again, we have

$$\int \sqrt{4-x^2}\,dx = 2\sin\theta\cos\theta + 2\theta + C \tag{1.9}$$
$$= x\sqrt{1-\frac{x^2}{4}} + 2\sin^{-1}\left(\frac{x}{2}\right) + C \tag{1.10}$$

(b) $\int_0^1 \frac{w}{(w^2+1)^{\frac{3}{2}}} dw$

Set

$$w = \tan\theta \quad \Rightarrow \quad dw = \sec^2\theta d\theta. \tag{1.11}$$

Limits:

$$w = 0 \quad \Rightarrow \quad \theta = 0 \tag{1.12}$$
$$w = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4} \tag{1.13}$$

Then

$$\int_0^1 \frac{w}{(w^2+1)^{\frac{3}{2}}} dw = \int_0^{\pi/4} \frac{\tan\theta}{\sec^3\theta}\ \sec^2\theta\,d\theta \tag{1.14}$$
$$= \int_0^{\pi/4} \sin\theta \tag{1.15}$$
$$= -\cos\theta\big|_0^{\pi/4} \tag{1.16}$$
$$= 1 - \frac{1}{\sqrt{2}} \tag{1.17}$$

Example 1.2 - More general forms

Calculate the following integrals using an appropriate trig substitution

(a)

$$\int \frac{1}{x^2+2x+5} dx \tag{1.18}$$

(b)

$$\int \frac{\sin^{-1} x}{x^2} dx \tag{1.19}$$

(a) $\int \frac{1}{x^2+2x+5} dx$

In completing the square, we put the integrand into a similar form to the previous examples.

$$I = \int \frac{1}{x^2+2x+5} dx = \int \frac{1}{(x+1)^2+4} dx \tag{1.20}$$

We notice the form "$a^2 + x^2$" on the denominator, so use the substitution

$$x + 1 = 2\tan\theta \quad \Rightarrow \quad dx = 2\sec^2\theta d\theta \tag{1.21}$$

Now

$$I = \int \frac{1}{4\tan^2\theta + 4} 2\sec^2\theta d\theta \tag{1.22}$$

$$= \frac{1}{2}\int d\theta \tag{1.23}$$

$$= \frac{1}{2}\theta + C \tag{1.24}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C \tag{1.25}$$

(b) $\int \frac{\sin^{-1} x}{x^2} dx$

Seeing $\sin^{-1}(x)$ in the integrand should make you think IBP. Set

$$u = \sin^{-1} x \quad \Rightarrow \quad du = \frac{1}{\sqrt{1-x^2}} dx \tag{1.26}$$

$$dv = \frac{1}{x^2} dx \quad \Rightarrow \quad v = -\frac{1}{x} \tag{1.27}$$

Then

$$I = \int \frac{\sin^{-1} x}{x^2} dx = -\frac{1}{x} \sin^{-1} x + \int \frac{1}{x\sqrt{1-x^2}} dx \tag{1.28}$$

The second integral requires a trig substitution. Let

$$x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta d\theta \tag{1.29}$$

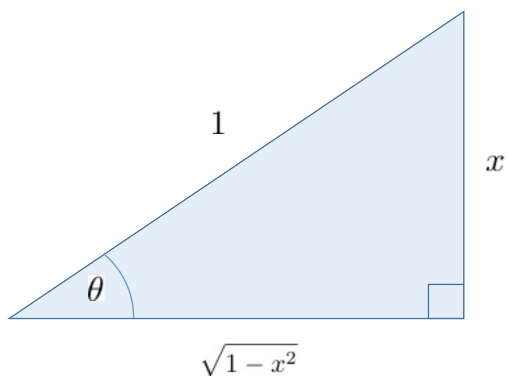
Then

$$\int \frac{1}{x\sqrt{1-x^2}} dx = \int \frac{1}{\sin\theta\cos\theta} \cos\theta\, d\theta \tag{1.30}$$

$$= \int \csc\theta d\theta \tag{1.31}$$

$$= -\ln|\csc\theta + \cot\theta| + C \tag{1.32}$$

To write this back in terms of x, consider the triangle that represents the situation:



We can then read off

$$\csc\theta = \frac{1}{x} \qquad \cot\theta = \frac{\sqrt{1-x^2}}{x}, \tag{1.33}$$

and so

$$I = -\frac{1}{x} \sin^{-1} x - \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C \tag{1.34}$$

Example 1.3 - Integrating rational functions

Evaluate

$$I = \int \frac{3x}{x^2 + x - 2} dx \tag{1.35}$$

We could solve this by completing the square and doing a trig substitution. However, if the denominator factorises nicely, then a partial fraction decomposition is probably faster...

$$\frac{3x}{x^2 + x - 2} = \frac{3x}{(x+2)(x-1)} \equiv \frac{A}{x+2} + \frac{B}{x-1} \tag{1.36}$$

And so

$$A(x-1) + B(x+2) \equiv 3x \tag{1.37}$$

Setting $x = 1$ gives $B = 1$. Setting $x = -2$ gives $A = 2$. Then

$$I = \int \left(\frac{2}{x+2} + \frac{1}{x-1} \right) dx \tag{1.38}$$

$$= 2 \ln |x+2| + \ln |x-1| + C \tag{1.39}$$

2 Areas Between Curves

- Set up integrals that represent particular areas.
- Integrate with respect to the horizontal or vertical coordinate where appropriate.

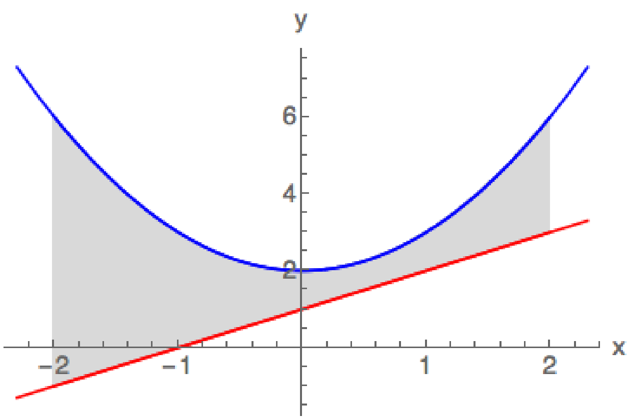
Example 2.1 - Vertically simple

Find the area between the curves

$$f(x) = x^2 + 2 \quad \text{and} \quad g(x) = x + 1 \tag{2.1}$$

over the interval $x \in [-2, 2]$.

A quick sketch shows that these two curves do not intersect and so the region we are evaluating is *vertically simple.*



Since $f(x) \geq g(x)$ on the interval $x \in [-2, 2]$, we can conclude that the desired area is

$$A = \int_{-2}^{2} (f(x) - g(x))\, dx \tag{2.2}$$

Since the interval of integration is symmetrical, we can break up the integrand into its even and odd parts to speed up the calculation:

$$A = \int_{-2}^{2} \left(x^2 - x + 1\right) dx \tag{2.3}$$

$$= \int_{-2}^{2} x^2 dx - \int_{-2}^{2} x dx + \int_{-2}^{2} 1 dx \tag{2.4}$$

$$= 2\int_{0}^{2} x^2 dx + \int_{-2}^{2} dx \tag{2.5}$$

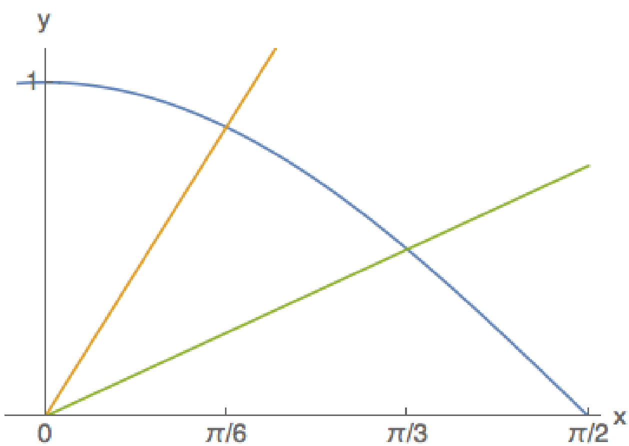
$$= \frac{16}{3} + 4 = \frac{28}{3} \tag{2.6}$$

Example 2.2 - Intersecting curves

Find the area enclosed by the curves

$$y_1 = \frac{3\sqrt{3}}{\pi}x, \quad y_2 = \frac{3}{2\pi}x, \quad y_3 = \cos x \tag{2.7}$$

as shown in the diagram below:



We see that the points of intersection occur at $x = \pi/6$ and $x = \pi/3$. The bounded area is vertically simple if broken up into two intervals:

$$A = \int_0^{\pi/6} (y_1 - y_2) dx + \int_{\pi/6}^{\pi/3} (y_3 - y_2) dx \tag{2.8}$$
$$= \int_0^{\pi/6} y_1 dx + \int_{\pi/6}^{\pi/3} y_3 dx - \int_0^{\pi/3} y_2 dx \tag{2.9}$$
$$= A_1 + A_2 - A_3 \tag{2.10}$$

Then we have

$$A_1 = \int_0^{\pi/6} \frac{3\sqrt{3}}{\pi} x\, dx \tag{2.11}$$
$$= \frac{3\sqrt{3}}{\pi}\ \frac{1}{2} x^2 \Big|_0^{\pi/6} \tag{2.12}$$
$$= \frac{3\sqrt{3}}{\pi}\ \frac{1}{2} \frac{\pi^2}{36} = \frac{\sqrt{3}\pi}{24}, \tag{2.13}$$

$$A_2 = \int_{\pi/6}^{\pi/3} \cos x dx \tag{2.14}$$
$$= \sin x \Big|_{\pi/6}^{\pi/3} \tag{2.15}$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{2} \tag{2.16}$$

and

$$A_3 = \int_0^{\pi/3} \frac{3}{2\pi} x dx \tag{2.17}$$
$$= \frac{3}{2\pi}\ \frac{1}{2} x^2 \Big|_0^{\pi/3} \tag{2.18}$$
$$= \frac{3}{2\pi}\ \frac{1}{2} \frac{\pi^2}{9} \tag{2.19}$$
$$= \frac{\pi}{12} \tag{2.20}$$

The enclosed area is then

$$A = \frac{\sqrt{3}\pi}{24} + \frac{\sqrt{3} - 1}{2} - \frac{\pi}{12} \tag{2.21}$$

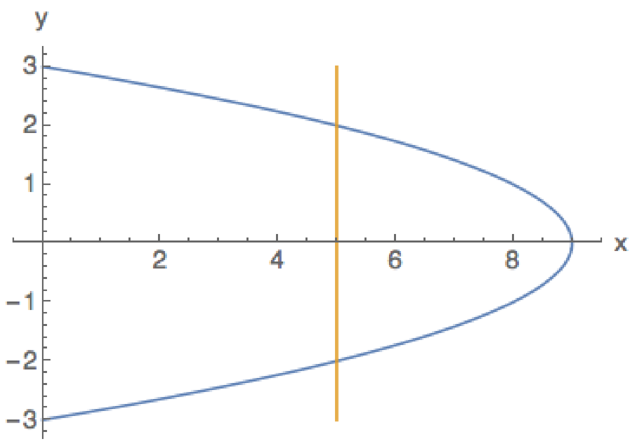
Example 2.3 - Integration over y

Find the area enclosed by the following curves using horizontal integration (integration along the y-axis)

$$x = 9 - y^2, \quad \text{and} \quad x = 5 \tag{2.22}$$

Verify your result with vertical integration.

Draw a sketch for visualisation and work out the points of intersection.



The curves intersect when

$$9 - y^2 = 5 \quad \Rightarrow \quad y = \pm 2 \tag{2.23}$$

Integrating over y, the area is then

$$A = \int_{-2}^{2} (9 - y^2 - 5) dy \tag{2.24}$$

$$= 2 \int_{0}^{2} (4 - y^2) dy \tag{2.25}$$

$$= 2 \left(4y - \frac{1}{3} y^3 \right) \Big|_0^2 \tag{2.26}$$

$$= 2 \left(8 - \frac{8}{3} \right) \tag{2.27}$$

$$= \frac{32}{3} \tag{2.28}$$

Alternatively, we could do the integral over x, adjusting the limits appropriately. By symmetry of the area

$$A = 2\int_5^9 \sqrt{9-x}\,dx \tag{2.29}$$

$$= 2\left(-\frac{2}{3}(9-x)^{3/2}\right)\Big|_5^9 \tag{2.30}$$

$$= 2\left(\frac{2}{3}4^{3/2}\right) \tag{2.31}$$

$$= \frac{32}{3}. \tag{2.32}$$