Examples 9
Integration Applications, Improper Integrals, Polar Coordinates and Complex Numbers

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The following are a set of examples to designed to complement a first-year calculus course. Learning objectives are listed under each section.*

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1 Integration Applications

- Calculate the length of curves
- Solve separable differential equations (with and without initial conditions)

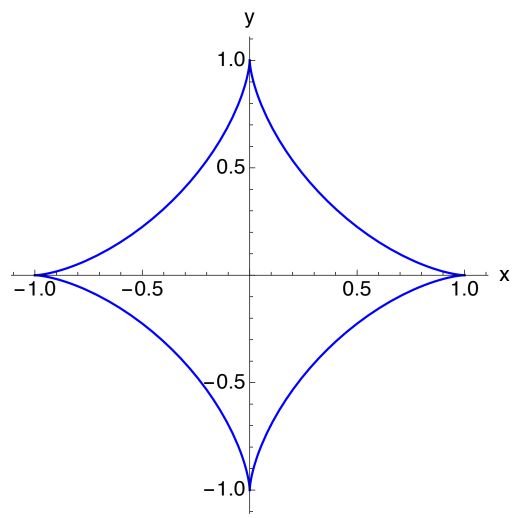
Example 1.1 - Arc length

Calculate the length of the astroid given by

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \tag{1.1}$$

Recall that the arc-length of $y = f(x)$ over the interval $[a, b]$ is given by

$$s = \int_a^b \sqrt{1 + f'(x)^2}\, dx \tag{1.2}$$



- The curve is symmetrical in transformations $x \to -x$ and $y \to -y$, therefore we can calculate the length in one quadrant and quadruple it.

- In the quadrant $\{(x, y) : x > 0, y > 0\}$, we have

$$y = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \tag{1.3}$$

upon rearranging (1.1) and taking the positive root.

- Then

$$\frac{dy}{dx} = \frac{3}{2}\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(-\frac{2}{3}x^{-\frac{1}{3}}\right) \tag{1.4}$$

$$= -x^{-\frac{1}{3}}\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \tag{1.5}$$

- The formula for arc length in this quadrant then gives

$$s = \int_0^1 \sqrt{1 + x^{-\frac{2}{3}} \left(1 - x^{\frac{2}{3}}\right)}\, dx \tag{1.6}$$

$$= \int_0^1 x^{-\frac{1}{3}} dx \tag{1.7}$$

$$= 3/2 \tag{1.8}$$

- Therefore the total length of the astroid is 6 units.

- ***Note:*** If it is easier to write the curve as a function of y, i.e. $x = g(y)$ then use

$$s = \int_{y_1}^{y_2} \sqrt{1 + g'(y)^2}\, dy \tag{1.9}$$

where $g'(y)$ is $\frac{dg}{dy}$ and y_1 and y_2 are y-values at the end points of the curve.

Example 1.2 - Separable differential equations

Solve the following differential equations

(a)
$$\frac{dy}{dx} = -xy \tag{1.10}$$

(b)
$$\frac{dT}{dt} = -k(T - 30^\circ), \qquad T(0) = 100^\circ \tag{1.11}$$

where k is a constant.

(a)
- No initial condition given, therefore we expect a family of solutions.
- Separating the variables gives

$$\int \frac{1}{y} dy = -\int x dx \tag{1.12}$$

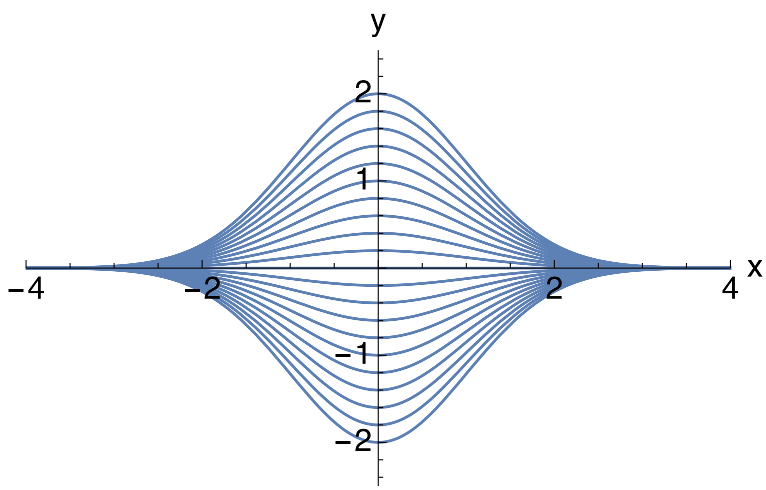
$$\Rightarrow \quad \ln y = -\frac{1}{2}x^2 + C \tag{1.13}$$

$$\Rightarrow \quad y = Ae^{-\frac{1}{2}x^2} \tag{1.14}$$

where we have introduced a new constant $A = e^C$.

- You can verify your solution by substituting it, and its derivative back into (1.10) and check that it is satisfied.

- Sketch of solutions for values of $A \in [-2, 2]$.



(b)

- This DE comes from Newton's Law of cooling: it represents the evolution of the temperature T of a body initially at $100°$, in a room of $30°$ degrees.

- Separate the variables:

$$\int \frac{1}{T-30}\, dT = \int -k\, dt \tag{1.15}$$

$$\Rightarrow \quad \ln(T-30) = -kt + C \tag{1.16}$$

$$\Rightarrow \quad T - 30 = Ae^{-kt} \qquad \text{where } A = e^C \tag{1.17}$$

$$\Rightarrow \quad T = Ae^{-kt} + 30 \tag{1.18}$$

- Now use the initial condition to find C:

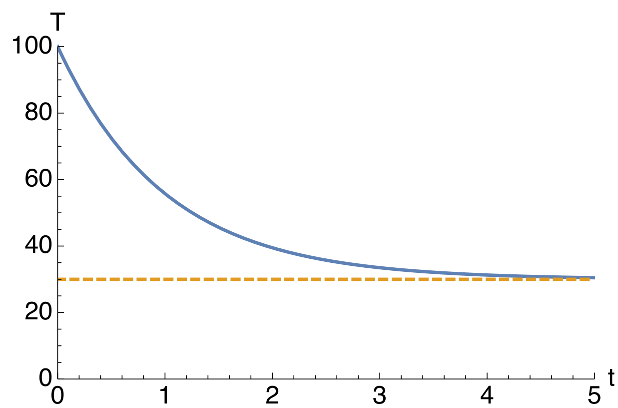
$$T(0) = A + 30 = 100 \tag{1.19}$$

$$\Rightarrow \quad A = 70 \tag{1.20}$$

- And so the evolution of the body's temperature satisfies

$$T = 70\, e^{-kt} + 30 \tag{1.21}$$

- Note that $\lim_{t\to\infty} T(t) = 30°$ as one would expect physically.



2 Improper Integrals

- Determine whether integrals with discontinuities in the integrand converge
- Evaluate these improper integrals

Example 2.1

Determine whether the following integrals converge. if they do, find their value.

(a)

$$\int_0^{\infty} x\, e^{-x^2} dx \tag{2.1}$$

(b)

$$\int_0^{2} \frac{1}{\sqrt{2-x}} dx \tag{2.2}$$

(a) - This is an improper integral due to the infinite upper boundary.

- We may write it as

$$I = \lim_{t\to\infty} \int_0^{t} x e^{-x^2} dx \tag{2.3}$$

- We may integrate using the substitution

$$u = x^2 \quad \Rightarrow \quad du = 2x\, dx \tag{2.4}$$

Then

$$I = \frac{1}{2} \lim_{t\to\infty} \int_0^{t^2} e^{-u} du \tag{2.5}$$

$$= \frac{1}{2} \lim_{t\to\infty} \left(-e^{-u}\right)\Big|_0^{t^2} \tag{2.6}$$

$$= \frac{1}{2} \lim_{t\to\infty} \left(1 - e^{-t^2}\right) \tag{2.7}$$

$$= \frac{1}{2} \tag{2.8}$$

(b) - This integral is improper since the integrand is not continuous at the upper boundary.

- Write it as

$$I = \lim_{t \to 2^-} \int_0^t \frac{1}{\sqrt{2-x}} dx \tag{2.9}$$

$$= \lim_{t \to 2^-} \left(-2(2-x)^{1/2} \right) \Big|_0^t \tag{2.10}$$

$$= \lim_{t \to 2^-} \left(-2\sqrt{2-t} + 2\sqrt{2} \right) \tag{2.11}$$

$$= 2\sqrt{2} \tag{2.12}$$

3 Polar Coordinates

- Convert between cartesian and polar coordinates
- Sketch curves expressed in polar coordinates

Example 3.1

Convert the following curve into Cartesian coordinates

$$r^2 = \frac{1}{\cos 2\theta} \tag{3.1}$$

Recall that the map between Cartesians and Polars may be expressed as

$$x = r\cos\theta, \quad y = r\sin\theta \tag{3.2}$$

- Using the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we have

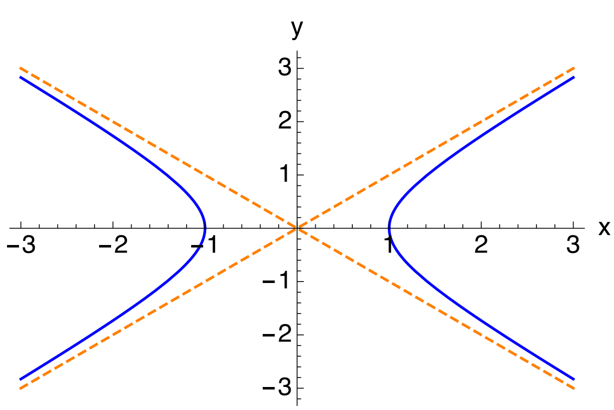
$$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta} \tag{3.3}$$

$$\Rightarrow \quad r^2(\cos^2\theta - \sin^2\theta) = 1 \tag{3.4}$$

$$\Rightarrow \quad x^2 - y^2 = 1 \tag{3.5}$$

which is the equation for a hyperbola.

- Note that the polar form is illuminating for a sketch - one can see straight away that there are asymptotes $\theta = \pm\pi/4$.



4 Complex Numbers

- Operations on complex numbers (multiplication / division etc.)
- Polar form of complex numbers
- DeMoivre's Theorem
- Integration using complex numbers
- Complex roots

Example 4.1

Write the following complex numbers in standard form:

(a)

$$\frac{1+j}{1-j} \tag{4.1}$$

(b)

$$\left(1+\sqrt{3}j\right)^3 \tag{4.2}$$

(a) We may simplify quotients of complex numbers by multiplying through by the complex conjugate of the denominator:

$$\frac{1+j}{1-j} = \frac{(1+j)^2}{(1-j)(1+j)} \tag{4.3}$$

$$= \frac{1+2j+j^2}{2} \tag{4.4}$$

$$= j \tag{4.5}$$

Alternatively we can use polar form (nice when multiplying and dividing complex numbers):

$$1+j = \sqrt{2}e^{\frac{\pi}{4}j}, \qquad 1-j = \sqrt{2}e^{-\frac{\pi}{4}j} \tag{4.6}$$

and so

$$\frac{1+j}{1-j} = \frac{\sqrt{2}e^{\frac{\pi}{4}j}}{\sqrt{2}e^{-\frac{\pi}{4}j}} = e^{\frac{\pi}{2}j} = j \tag{4.7}$$

(b) Now that there's a lot of multiplication involved, we should definitely convert to polars:

$$r = \sqrt{1+3} = 4, \qquad \tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3} \qquad (\theta \text{ in 1st quadrant}) \tag{4.8}$$

Then

$$1 + \sqrt{3}j = 2\, e^{\frac{\pi}{3} j} \tag{4.9}$$

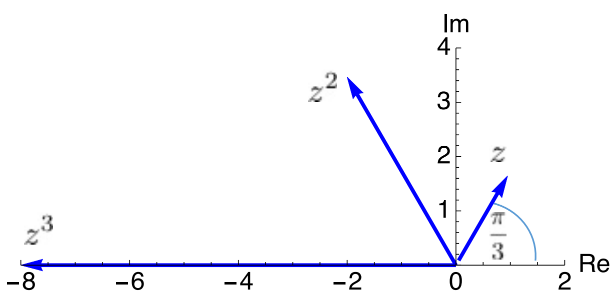
and so

$$(1+\sqrt{3}j)^3 = \left(2\, e^{\frac{\pi}{3} j}\right)^3 \qquad \text{(De Moivre's Theorem)} \tag{4.10}$$

$$= 8\, e^{\pi j} \tag{4.11}$$

$$= -8 \tag{4.12}$$

This is clear geometrically in the complex plane recalling that upon multiplying complex numbers, the moduli get multiplied and the arguments get added:



Example 4.2

Use complex numbers to evaluate the following integral:

$$\int e^{2x} \cos 3x \, dx \tag{4.13}$$

- Note that $\cos 3x$ is the real part of e^{3jx} so

$$\int e^{2x} \cos 3x \, dx = \int e^{2x} \Re\left(e^{3jx}\right) \, dx = \Re\left(\int e^{(2+3j)x} \, dx\right). \tag{4.14}$$

- Evaluating the complex integral gives

$$\int e^{(2+3j)x} \, dx = \frac{1}{2+3j} e^{(2+3j)x} + \bar{C} \tag{4.15}$$

$$= \frac{2-3j}{13} e^{2x} \left(\cos 3x + j \sin 3x\right) + \bar{C} \tag{4.16}$$

where $\bar{C}$ is a complex constant.

- Taking the real part gives us back our original integral:

$$\int e^{2x} \cos 3x \, dx = \frac{2}{13} e^{2x} \cos 3x + \frac{3}{13} e^{2x} \sin 3x + C \tag{4.17}$$

where $C = \Re(\bar{C})$.

Example 4.3

Find all the fourth roots of -16

- Write in polar form:

$$r = 16, \ \ \theta = \pi \quad \Rightarrow \quad -16 = 16e^{\pi j} \tag{4.18}$$

- However since the polar form is invariant to adding multiples of 2π to θ we may write

$$-16 = 16e^{j\pi} = 16e^{j(\pi+2k\pi)} \tag{4.19}$$

  for integer values of k.

- Taking the fourth root gives

$$(-16)^{\frac{1}{4}} = 16^{\frac{1}{4}} e^{j\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)} \tag{4.20}$$

$$= 2e^{j\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)} \tag{4.21}$$

- Now run through four consecutive values for k.

$$k = 0 \quad \text{gives} \quad \omega_0 = 2e^{\frac{\pi}{4}j} = \sqrt{2}(1+j) \tag{4.22}$$

$$k = 1 \quad \text{gives} \quad \omega_1 = 2e^{\frac{3\pi}{4}j} = \sqrt{2}(-1+j) \tag{4.23}$$

$$k = 2 \quad \text{gives} \quad \omega_2 = 2e^{\frac{5\pi}{4}j} = \sqrt{2}(-1-j) \tag{4.24}$$

$$k = 3 \quad \text{gives} \quad \omega_3 = 2e^{\frac{7\pi}{4}j} = \sqrt{2}(1-j) \tag{4.25}$$