Lecture 6.

Today's topics:
- Modelling (continued)
- exponential functions

Read Ch 2.3.
Ex. 2.3.1 - 2.3.4.
EoL 6.

Recall: tree model

$D(t) = a\sqrt{t+b}$ from data, found $a=10$, $b=16$

Fitted model $D(t) = 10\sqrt{t+16}$.

Algebraic check: $D(0) = 10\sqrt{16} = 40$ cm. ✓
$D(20) = 10\sqrt{36} = 60$ cm. ✓

Follow up qs.

1. If $t=0$ csp. to 1997, when did tree start growing?

   ie. Find t such that $D(t) = 0$

   From model: $0 = 10\sqrt{t+16} \Rightarrow t = -16$, 1981. $1997-16 = 1981$

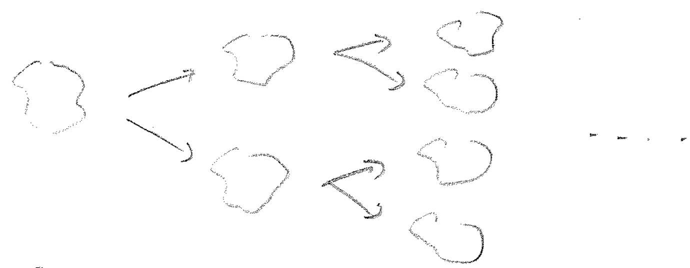
2. In what year does diameter reach 70cm?

$$70 = 10\sqrt{t+16} \Rightarrow \sqrt{t+16} = 7$$
$$\Rightarrow t+16 = 49$$
$$\Rightarrow t = 33,$$

$1997 + 33 = 2030$

Modelling Cellular Growth

Human body starts as single cell (zygote) which undergoes serial duplication - assume once per day.



day (d)	0	1	2	3	4
# cells N	1	2	4	8	16

Model 1

Cubic model fit to data gives

$$N_1(d) = \frac{d^3}{3} - d^2 + \frac{8d}{3}$$

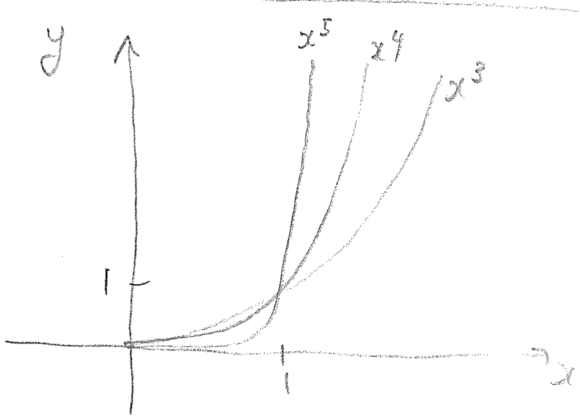
(check agrees with data).

Test model for larger times:

Newborn baby $N \approx 1 \times 10^{12}$ cells.

Model N_1 hits 1×10^{12} after $d = 14{,}423$ days (40 years!)

Higher degree polynomial? Eg. x^4, x^5?



Nope: Serial duplication eventually grows faster than any polynomial.

Model 2 - Exponential function

$N_2(d) = 2^d$ — check satisfies data

$N_2(1)=2$, $N_2(2)=4$ etc.

$N_2(40) \simeq 1\times 10^{12}$ cells.

40 days to reach newborn site - need exponential functions to describe this behaviour

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Review: Laws of exponents

Tip: write roots as fractional exponents. Eg.
$\sqrt[3]{a\sqrt{b}} = (ab^{\frac{1}{2}})^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{6}}$.

For $a, b > 0$, $x, y \in \mathbb{R}$

1. $a^{x+y} = a^x a^y$ Eg/ $2^{3+x} = 2^3 . 2^x = 8(2^x)$
2. $a^{x-y} = \dfrac{a^x}{a^y}$ Eg/ $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$
3. $(a^x)^y = a^{xy}$ Eg/ $(2^x)^3 = 2^{3x} = (2^3)^x = 8^x$
4. $(ab)^x = a^x b^x$ Eg/ $(2a)^2 = 2^2 a^2 = 4a^2$.

General exponential function.

$f(x) = a^x$, $a > 0$. (not defined ow).

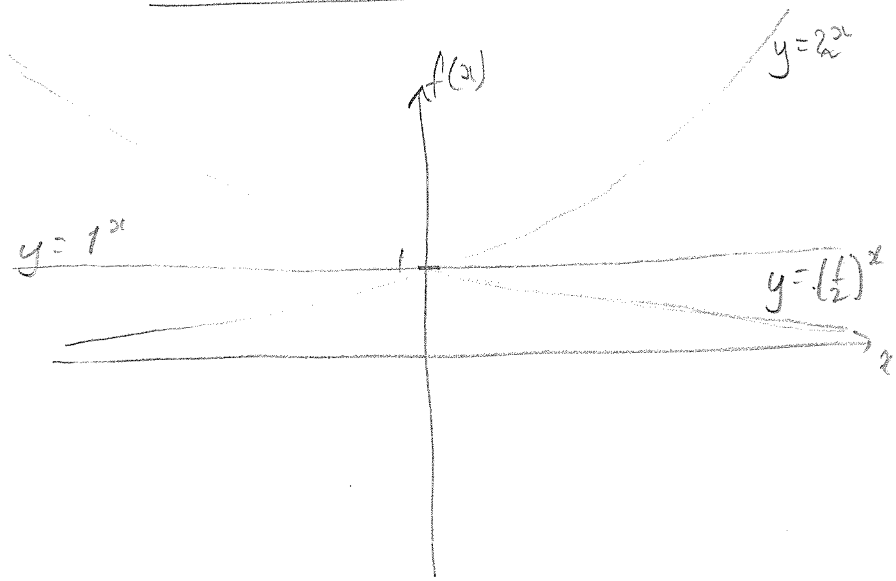
base, power/exponent

$f(x)$ is a transcendental fn..

- Some output values cannot be computed using algebraic operations

- $a^3 = a\times a\times a$ ✓ $a^\pi = ?$

Graphing $f(x) = a^x$.



$a=2$: $y = 2^x$.

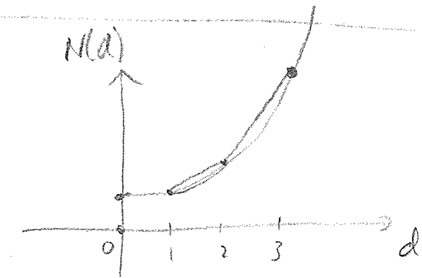
$a = \frac{1}{2}$ $y = \left(\frac{1}{2}\right)^x = 2^{-x}$

$a=1$ $y = 1^x = 1$

$\mathbb{D} = \mathbb{R}$.

$\mathbb{E} = \{x \in \mathbb{R} : x > 0\}$.

Interpreting Exponential Models



$$N(d) = 2^d.$$

$$N(d+1) = 2^{d+1} = 2 \times 2^d = 2 \times N(d)$$

Growth of population over single day $= N(d+1) - N(d)$

$= 2N(d) - N(d)$

$= N(d)$

Seems that growth rate $\propto N(d)$.

Key property of exponential functions.

$$f(x) = a^x$$

$\dfrac{df}{dx} \propto f$: rate in change of f is proportional to f

$= (kf)$

↳ will derive later in course!

The natural exponential

$$f(x) = a^x$$

There is a base where $\overbrace{\text{growth rate}}^{\text{slope / gradient / derivative}} = f$ exactly.

$a = e = 2.7182818 \ldots$