Lecture 15.

Today's topics:

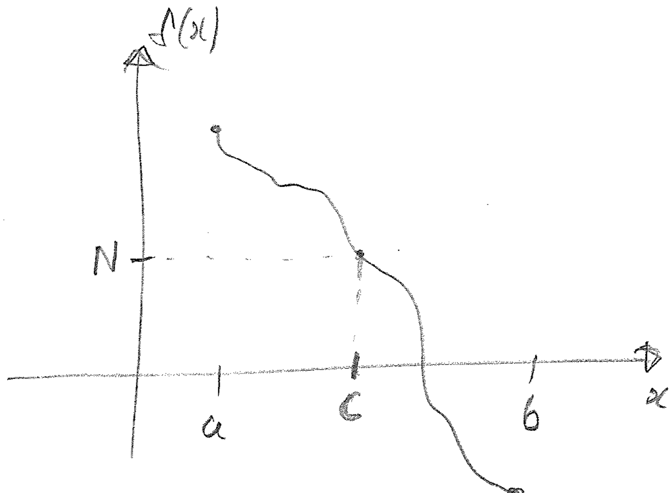
- The bisection algorithm.
- The derivative!

Read Ch 4.1, 4.2.
+ examples.

Ex. 3.7.4, 3.7.5.
(bisec. alg.).

Midterm today 5.30pm
– good luck!

Recall: The IVT.



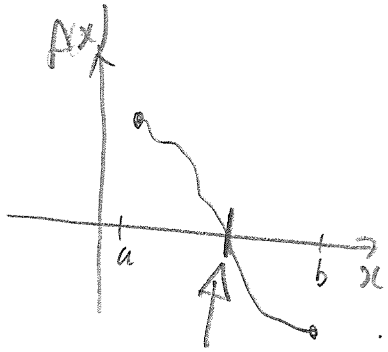
If:

1) f continuous on $[a,b]$.
2). $N \in (f(a), f(b))$

Then:

Can find $c \in [a,b]$ such that $f(c) = N$

Root finding



From IVT, if

1) f continuous on $[a,b]$
2) $f(a) > 0$, $f(b) < 0$ (or vice-versa)

Then:

can find c $\in [a,b]$ such that $f(c) = 0$.

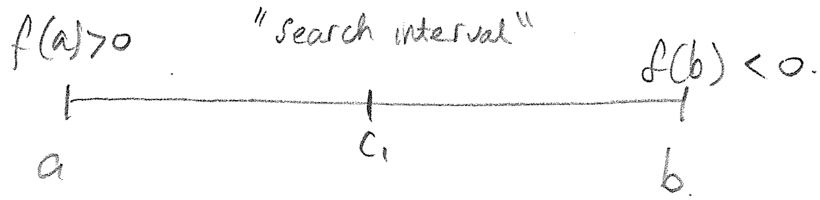
i.e. a root exists. in $[a,b]$.

The Bisection Algorithm

"division into equal parts".

→ allows us to approximate the root, using IVT as a guide.

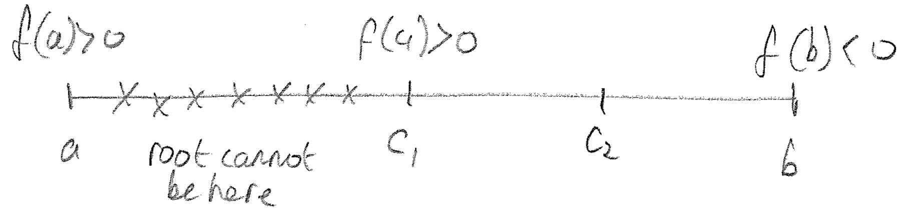
Suppose f has exactly 1 root in (a, b).
Assume $f(a) > 0$, $f(b) < 0$.



→ Test the midpoint, $c_1 = \frac{a+b}{2}$

$f(c_1) =$ positive or negative ?

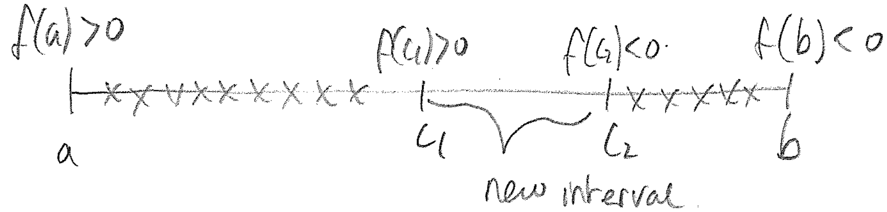
Suppose $f(c_1) > 0$.



root must belong to (c_1, b) as $f(x)$ changes sign only once.

→ Repeat for next midpoint, $c_2 = \frac{c_1 + b}{2}$

→ Say $f(c_2) < 0$

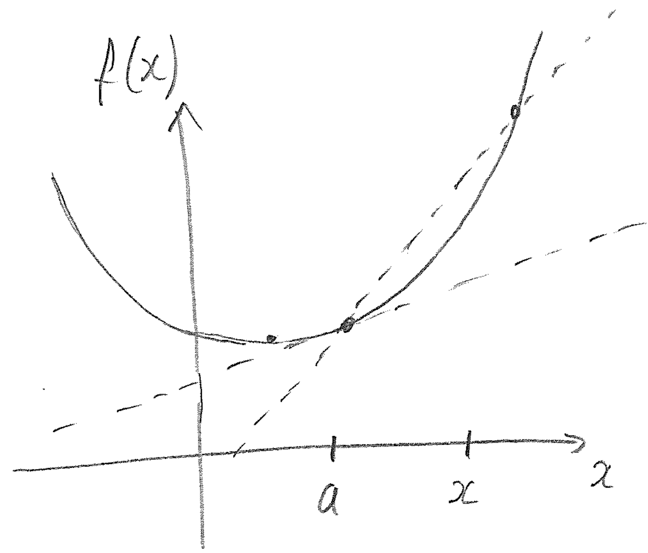


→ repeat until desired precision is reached.

Eg. 

sol = 0.15 ± 0.05.

The Derivative at a point



Gradient from
$(a, f(a))$ to $(x, f(x))$
is

$$\frac{f(x) - f(a)}{x - a} = \frac{\text{"rise"}}{\text{run}}.$$

Take limit as $x \to a$ to get

derivative $\iff$ instantaneous rate of change $\iff$ slope of tangent line

at the point $x = a$.

Notation:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

"the derivative of $f(x)$ at $x = a$"

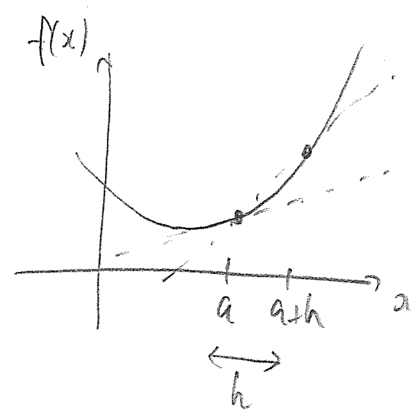
Note: this limit may not exist, in which case the derivative does not exist.

Alternate (and useful) form

Set $x = a + h$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Sometimes easier to work with a limit as $h \to 0$.



Example 1.

Let $f(x) = 2x^2 + x$

Find $f'(a)$ using defn of a derivative at a point.

Approach #1.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{2(a+h)^2 + a + h - (2a^2 + a)}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 + 4ah + h}{h}$$

$$= \lim_{h\to 0} (2h + 4a + 1)$$

$$= 4a + 1.$$

Approach #2

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x\to a} \frac{2x^2 + x - (2a^2 + a)}{x - a}$$

$$= \lim_{x\to a} \frac{2(x^2 - a^2) + x - a}{x - a}$$

$$= \lim_{x\to a} \frac{2(x+a)(x-a) + x - a}{x - a}$$

$$= \lim_{x\to a} 2(x+a) + 1$$

$$= 4a + 1$$

(a bit harder, but both approaches always work!)

Example 2

Find $f'(1)$ for $f(x) = \sqrt{x}$ using the defn of the derivative.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Sub $a=1$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} \qquad \text{“} \frac{0}{0} \text{”}$$

When to process limit is important.
Here, $\frac{0}{0}$ is inconclusive, so must manipulate the expression before evaluating the limit.

$$f'(1) = \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \times \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$$

"conjugate method".

$$= \lim_{h \to 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2}$$

now process the limit as it is conclusive.