Lecture 22

Read Ch 5.4.1, 5.4.2

Ex 5.4.1 - 5.4.7

(EoL 20)

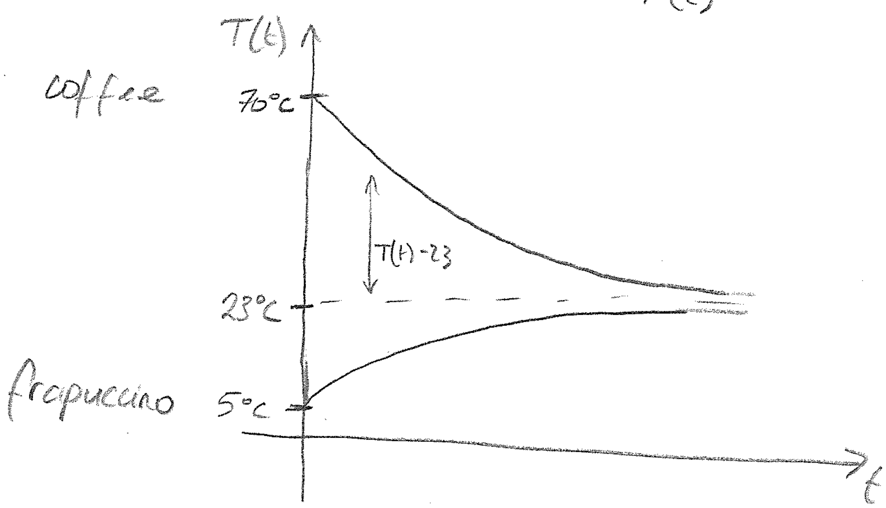
Today's topics :-

- Exponential convergence
- Linear approximations

Exponential Convergence.

$$y(t) = y_0 e^{kt} \begin{cases} \to \infty & \text{as } t \to \infty \text{ for } k > 0 \text{ (growth)} \\ \to 0 & \text{as } t \to \infty \text{ for } k < 0 \text{ (decay)} \end{cases}$$

What about the temperature of a cooling coffee?
$T(t)$



- convergence to a non-zero value
- $T(t) \to 23°C$ (room temp)
- exponential decay of $T(t) - 23$.

When the difference between a (current) quantity and the value to which it converges decays exponentially, we have exponential convergence.

Example: Newton's law of cooling

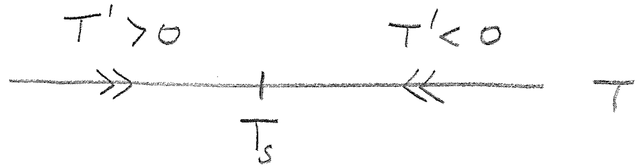
$T(t)$: temperature of object at time t

T_s : temp. of surroundings (constant)

Then $$\frac{dT}{dt} = k(\underbrace{T - T_s}_{})$$

cooling constant
$k < 0$

difference in temp
between object & surroundings.



temp. approaches T_s.

Solution?

- Trick: introduce variable $y(t) = T(t) - T_s$ (difference in temp).
- $\frac{dy}{dt} = \frac{dT}{dt}$ (since T_s constant)

$\Rightarrow \frac{dy}{dt} = ky$ (exponential decay in temp. difference)

$\Rightarrow y(t) = y_0 e^{kt}$, $\quad y_0 = T_0 - T_s$.

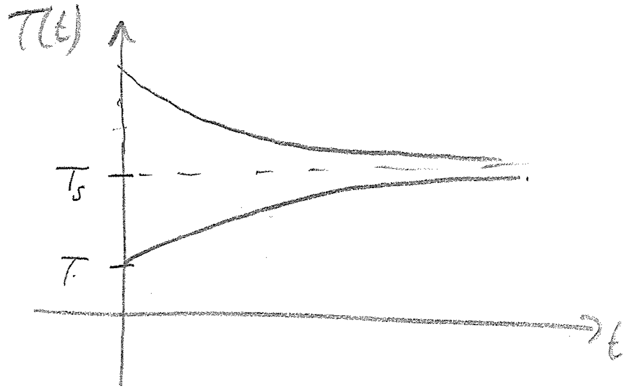
initial temp.

$\Rightarrow T(t) - T_s = (T_0 - T_s)e^{kt}$

$\Rightarrow T(t) = T_s + (T_0 - T_s)e^{kt}$.

Note: $T(0) = T_0$.

$T(t) \to T_s$ as $t \to \infty$.



exponential convergence

(exp. decay in $T - T_s$).

Ex/ Suppose a bird egg is kept at 35°C by parent bird. While parent goes foraging, egg cools to 30°C in the 18°C air, when left for 1 hour.

If egg must remain above 25°C for survival, how long can parent safely forage?

Important info: Let temperature $= T(t)$

$T_0 = 35°C$

$T_s = 18°C$

$T(1) = 30°C$

$T(t^*) = 25°C$

find

'NLC'

$$\frac{dT}{dt} = k(T - T_s)$$

Let $y(t) = T(t) - T_s$.

$$\Rightarrow \frac{dy}{dt} = ky, \quad y(t) = y_0 e^{kt}$$

$y_0 = y(0) = T(0) - T_s = 35 - 18 = 17°C$.

$y(1) = T(1) - T_s = 30 - 18 = 12°C$

$y(1) = 12$

$\Rightarrow y_0 e^{k \cdot 1} = 12$

$\Rightarrow 17 e^{k} = 12$

$\Rightarrow k = \ln\left(\frac{12}{17}\right)$.

$$\Rightarrow y(t) = 17 e^{\ln\left(\frac{12}{17}\right)t}$$

$$\Rightarrow T(t) = 18 + 17 e^{\ln\left(\frac{12}{17}t\right)}$$

Let t^* be time when T reaches $25°C$.

$$T(t^*) = 25$$

$$\Rightarrow 18 + 17e^{\ln(\frac{12}{17})t^*} = 25$$

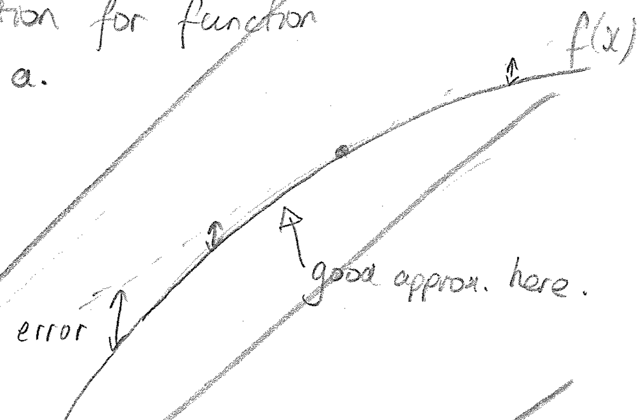
$$\Rightarrow e^{\ln(\frac{12}{17})t^*} = \frac{7}{17}$$

$$\Rightarrow t^* = \frac{\ln(\frac{7}{17})}{\ln(\frac{12}{17})} \approx 2.55 \text{ hours.}$$

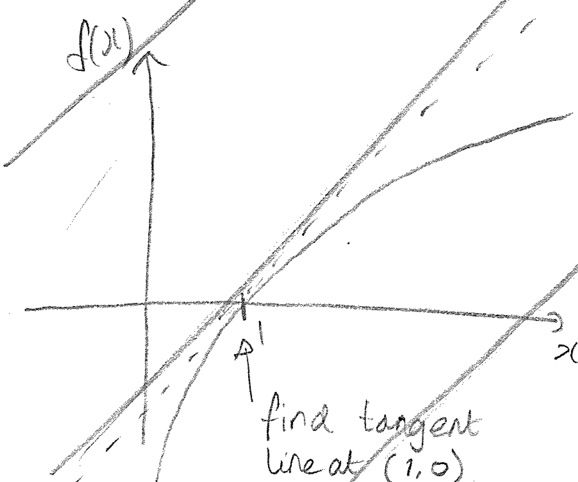
$\therefore$ parent can leave the egg for at most 2.55 hours.

Linear Approximations

- The tangent line of $f(x)$ at $x=a$ can serve as an approximation for function values at points close to a.



Eg/. $f(x) = \ln x$



$$y - y_0 = m(x - x_0)$$

$m = f'(1)$, $x_0 = 1$, $y_0 = 0$.
$= \frac{1}{1} = 1$.

$$y = x - 1.$$

Linear Approximations

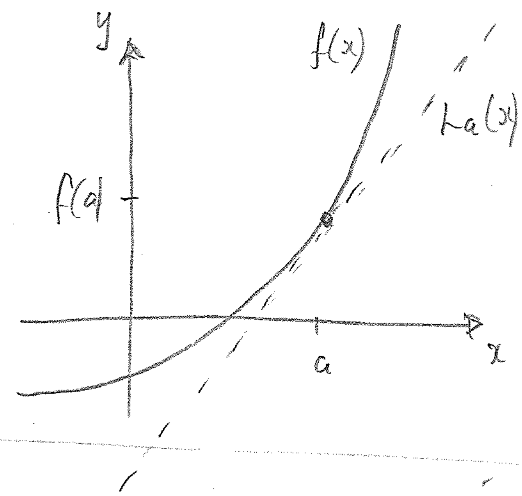
Recall:

The tangent line of $f(x)$ at $x=a$ $\Longleftrightarrow$ "Linearization" of $f(x)$ at $x=a$ $\Longleftrightarrow$ $L_a(x)$

often omit the 'a'

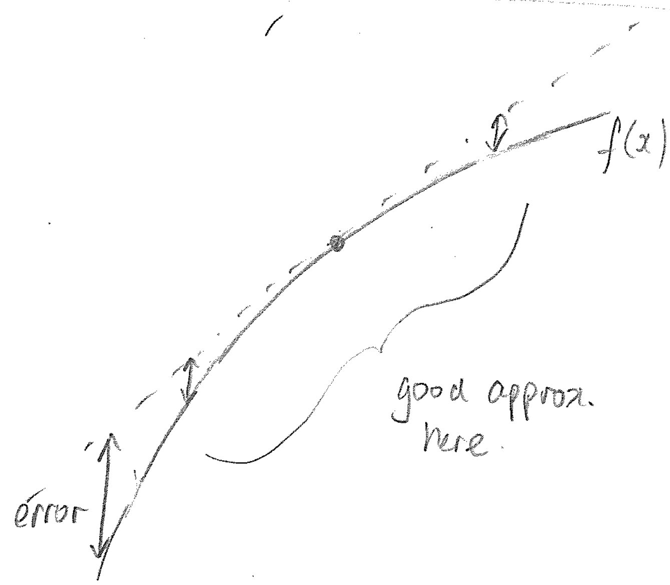
$$L_a(x) = f(a) + (x-a)f'(a)$$

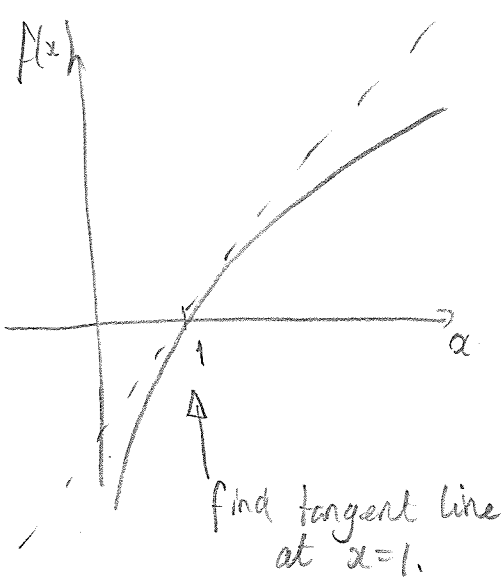
- $L_a(x)$ serves as a good approximation to $f(x)$ for x close to a.



Example

$f(x) = \ln x$. Approximate $\ln(1.1)$.



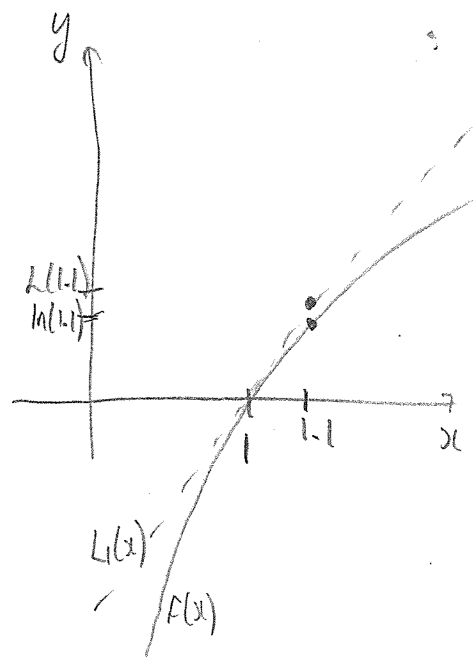


$$\begin{aligned} L_1(x) &= f(1) + (x-1)f'(1) \\ &= 0 + (x-1)\left(\frac{1}{1}\right) \\ &= x-1 \end{aligned}$$

Approximate $\ln(1.1)$ using $L_1(1.1)$.

$$\left.\begin{array}{l} L_1(1.1) = 1.1 - 1 = 0.1 \\ \ln(1.1) \simeq 0.095 \end{array}\right\} \text{pretty good.}$$

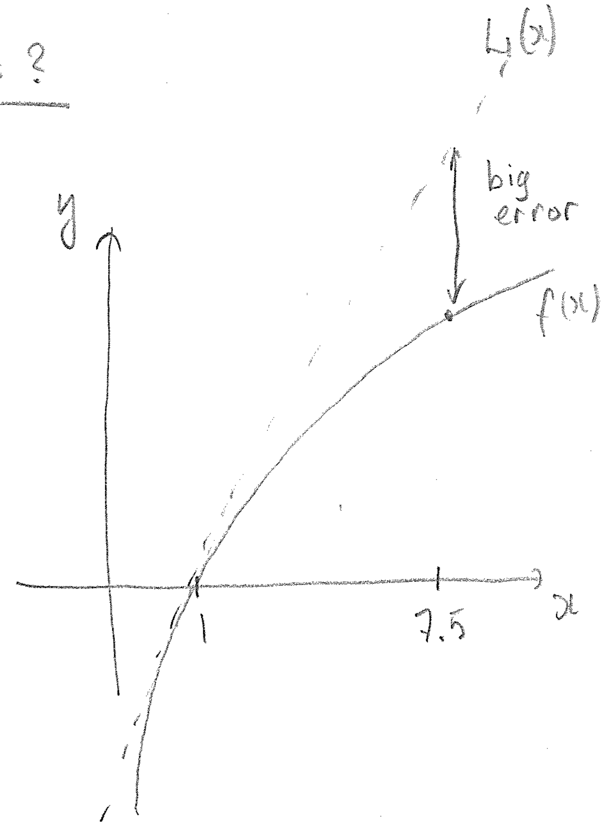
↑
(calc).



Further away from tangent point?

Eg/ Approximate $\ln(7.5)$.

$$\left.\begin{array}{l} L_1(7.5) = 6.5 \\ \ln(7.5) \simeq 2.015 \end{array}\right\} \text{oh dear...}$$



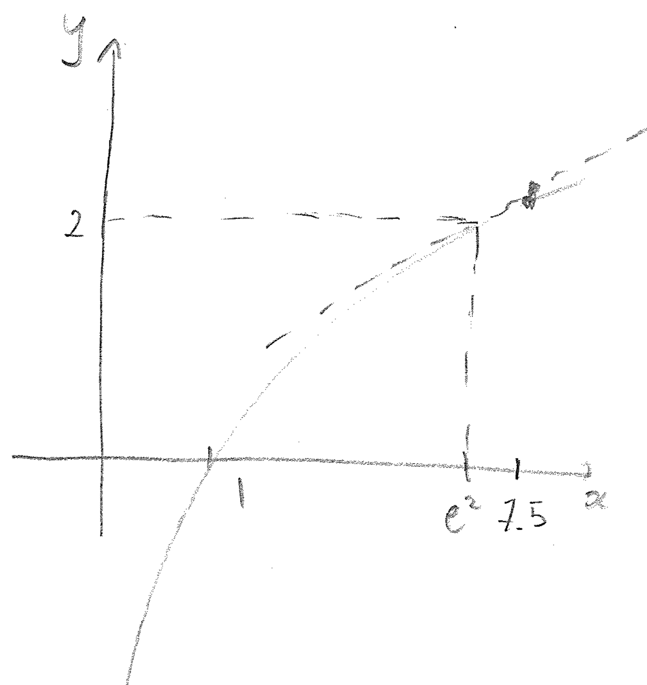
Workaround ?

- Choose the tangent point so that
  1) it is close to the given function input
  2) it is a 'nice' value to put into function.

ln(7.5).

1) $e^2 \simeq 7.39$ is close.
2) $\ln(e^2)$ is nice.

$$L_{e^2}(x) = f(e^2) + (x - e^2) f'(e^2)$$
$$= \ln(e^2) + (x - e^2)\frac{1}{e^2}$$
$$= 2 + \frac{x}{e^2} - 1$$
$$= \frac{x}{e^2} + 1.$$



$$L_{e^2}(7.5) = \frac{7.5}{e^2} + 1 \simeq 2.0150$$
$$\ln(7.5) \simeq 2.0149$$

much better.