Lecture 31

Today's topics:

- Riemann sums
- The definite integral

Read Ch 6.2

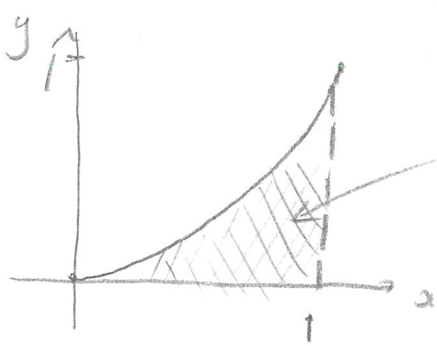
Example 6.1.7

EoL 27-28 (due Fri)

Biweekly 5 for practice

Applying the "rectangle strategy"

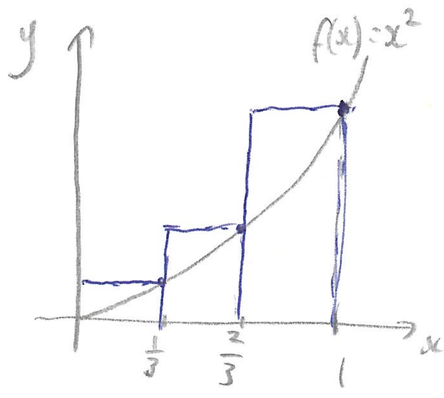
Goal: find the area beneath $y = x^2$ from $x = 0$ to $x = 1$



Let's approximate this with rectangles.

→ make top-right corner touch curve.

Attempt #1: 3 rectangles



height, determined by $f(x) = x^2$

$$\text{Area} = \underbrace{\left(\tfrac{1}{3}\right)}_{\text{base of rectangles}}\left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{3}\right)\left(1\right)^2$$

base of rectangles

$$= \sum_{i=1}^{3} \left(\tfrac{1}{3}\right)\left(\tfrac{i}{3}\right)^2$$

← Riemann sum (approx. area using rectangles)

$$\sum_{i=1}^{3}\left(\frac{1}{3}\right)\left(\frac{i}{3}\right)^2 = \frac{1}{27}\sum_{i=1}^{3} i^2$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$$

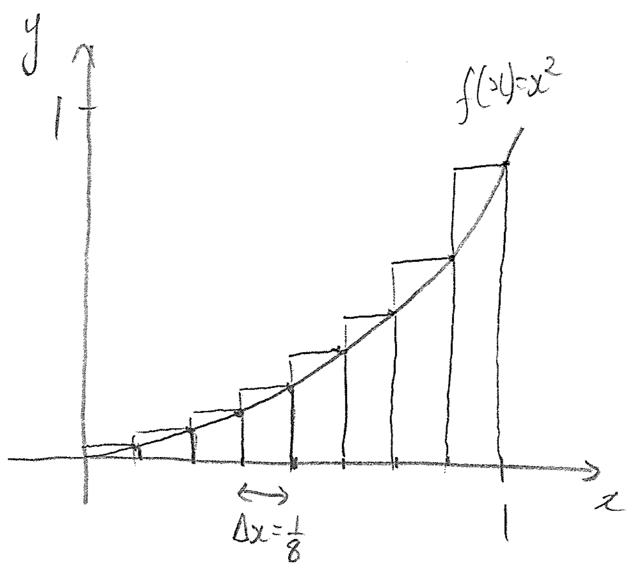
$$= \frac{1}{27}\left[\frac{1}{6}(3)(4)(7)\right]$$

$$= \frac{1}{27}\left[14\right] = \frac{14}{27} \approx 0.52$$

→ True area $= \frac{1}{3}$

→ Approximation is a big overestimate as expected.

Attempt #2: 8 rectangles.



$$\text{Area} = \sum_{i=1}^{8} \underbrace{\left(\frac{1}{8}\right)}_{\text{base}} \overbrace{\left(\frac{i}{8}\right)^2}^{\text{height, } f(x)}$$

(The x values are given by $x_i = \frac{i}{8}$, $i = 1, 2, \ldots 8$).

$$\text{Area} = \frac{1}{8^3}\sum_{i=1}^{8} i^2$$

$$= \frac{1}{512}\cdot\frac{1}{6}(8)(9)(17)$$

$$= \frac{204}{512} \approx 0.398$$

→ Gets us closer... we need more rectangles!

Attempt #3 : Infinite rectangles!

Notice a pattern in area calculation.

For "n" rectangles

$$\text{Area} = \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^3}\,\frac{1}{6}\,n(n+1)(2n+1).$$

Give it all the rectangles!

$$\text{Area} = \lim_{n\to\infty} \frac{1}{n^3}\,\frac{1}{6}n(n+1)(2n+1) = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2}$$

$$= \lim_{n\to\infty} \frac{2+3/n+1/n^2}{6}$$

$$= \frac{2}{6} = \frac{1}{3} \qquad \longleftarrow \text{exact!}$$

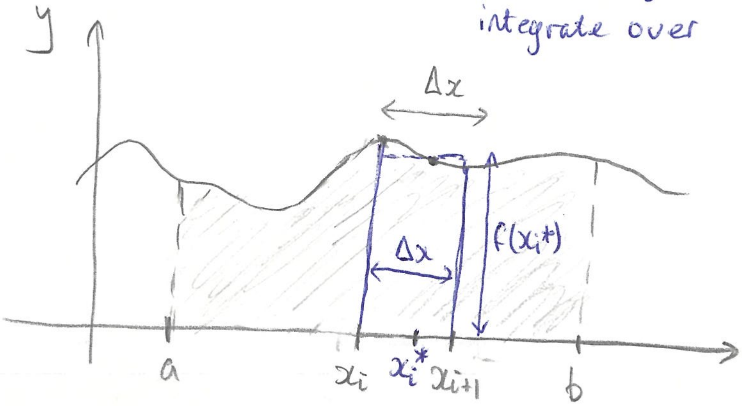
→ Infinitely many rectangles yields exact area computation.

→ This leads to the following definition:

Definition: The definite integral.

→ The definite integral of f from a to b is

$$\underset{\text{lower bound}}{\int_a}^{\overset{\text{upper bound}}{b}} f(x) \underbrace{dx}_{\substack{\text{symbol indicating} \\ \text{which variable to} \\ \text{integrate over}}} = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\text{height}} \underbrace{\Delta x}_{\substack{\text{width of} \\ \text{rectangles}}}$$



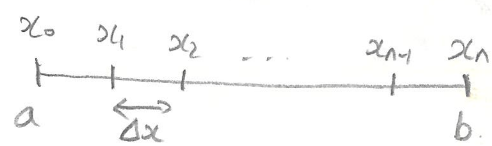
$x_i^* \in [x_i, x_{i+1}]$.

→ Previously we set the height of the rectangles to be f value on right-hand edge.

→ In limit as $n\to\infty$, can use any $x_i^* \in [x_i, x_{i+1}]$.

Notes: $\Delta x = \dfrac{b-a}{n}$, width of rectangles is interval $[a,b]$ divided by n.

$(i=0,\dots,n)$ $\quad x_i = a + i\Delta x$



Example

Using the definition of the integral, compute $\int_1^4 (x+2)\,dx$.

Set up:

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}, \qquad f(x) = x+2.$$

$$x_i = a + i\Delta x = 1 + i\frac{3}{n} = 1 + \frac{3i}{n}.$$

$$\int_1^4 (x+2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

(typically set $x_i^* = x_i$)

Evaluate:

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[1+\frac{3i}{n}+2\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(3+\frac{3i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\sum_{i=1}^{n} 3 + \frac{3}{n}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[3n + \frac{3}{n}\frac{1}{2}n(n+1)\right]$$

$$= \lim_{n\to\infty} \left(9 + \frac{9(n+1)}{2n}\right)$$

$$= 9 + \frac{9}{2}\lim_{n\to\infty} \frac{n+1}{n}$$

$$= 9 + \frac{9}{2} = \frac{27}{2}$$

(Long and tedious – let's see how anti-derivatives can help us!)

Next time:

→ How are anti-derivatives related to the definite integral?

→ The Fundamental Theorem of Calculus.