Lecture 32

Today's topics:

- The Fundamental Theorem of Calculus.
- Computing definite integrals using antiderivatives.

Read Ch 6.3

Ex 6.2.1 - 6.2.6
6.3.1 - 6.3.9

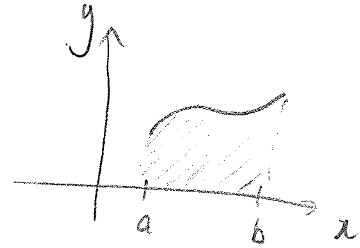
EoL 30 (EoL 29 next lec)

EoL 23-28 due 4pm

Monday's tutorial - Project 3 help.

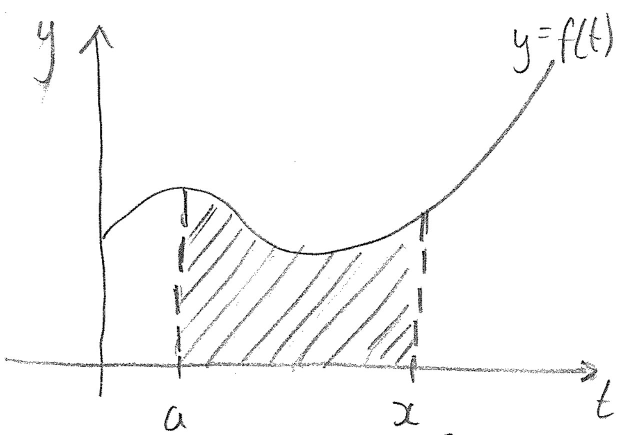
Last time: The definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$



→ long and tricky to compute
→ computing antiderivatives is faster
→ how are antiderivatives and area related?

The Area Function

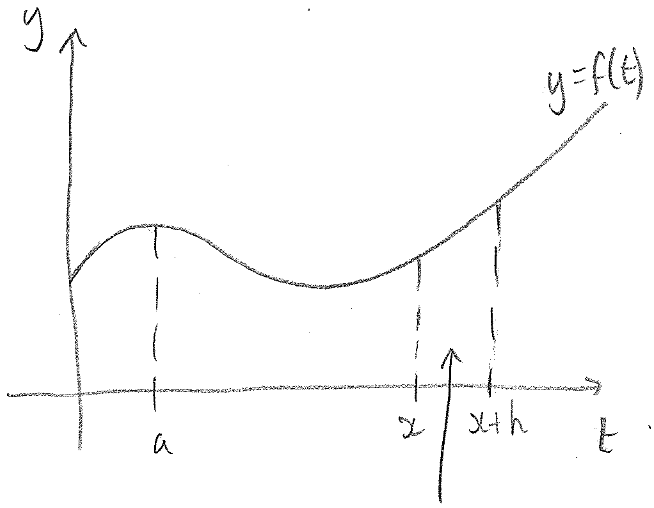


← this upper bound is a variable.

→ Area under curve from 'a' to 'x' is

$$A(x) = \int_a^x f(t)\,dt.$$

The Rate of Change in Area.



approximately a rectangle for h small

→ Introduce a small change to x

→ Gives a small change in area

→ Old area $= A(x)$
New area $= A(x+h)$

→ Change in area $\approx h.f(x)$

So $\quad A(x+h) - A(x) \approx h f(x)$

$$\Rightarrow \quad \frac{A(x+h) - A(x)}{h} \approx f(x)$$

Take limit as $h \to 0$

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \frac{dA}{dx}.$$

$\Rightarrow$ $A(x)$ is an antiderivative of $f(x)$!

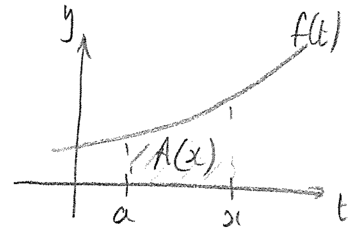
The Fundamental Theorem of Calculus (FTOC)

Given a function f that is continuous over the interval $[a,b]$, define $A(x)$ as

$$A(x) = \int_{t=a}^{t=x} f(t)\,dt.$$

Then, on the interval (a,b),

$$A'(x) = f(x).$$

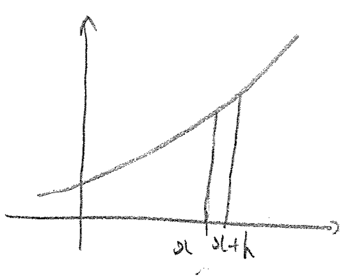


Note: value of a doesn't affect $A'(x)$.

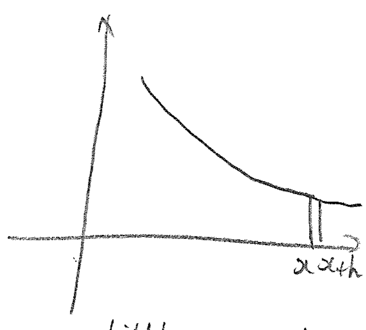
What is the FTOC telling us?

$$A'(x) \quad = \quad f(x)$$

rate in change of area at x = height of curve at x



big gain at x



Little gain at x.

Can it help us integrate?

$A'(x) = f(x) \Rightarrow A$ is an antiderivative of f.

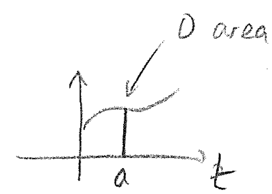
Any antiderivative of f can then be written

$$F(x) = A(x) + C, \quad \text{for some constant } C.$$

$$\Rightarrow \quad F(b) - F(a) = [A(b)+C] - [A(a)+C]$$
$$= A(b) - A(a)$$
$$= \int_a^b f(t)\,dt - \cancel{\int_a^a f(t)\,dt}$$

0.

This gives another form of the FTOC.



The FTOC (ver II)

Given a function f that is continuous over $[a, b]$,

Definite integral $\longrightarrow$ $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

where $F(x)$ is an antiderivative of $f(x)$.

$\hookrightarrow$ doesn't matter which one ($+C$ cancels)

Back to integral example

$$\int_1^4 (x+2)\,dx \qquad f(x) = x+2.$$
$$F(x) = \tfrac{1}{2}x^2 + 2x + C.$$

$= F(4) - F(1)$

$= \frac{1}{2}(16) + 2(4) + C - \left[\frac{1}{2} + 2 + C\right]$

$= 8 + 8 - \frac{1}{2} - 2$

$= 16 - \frac{5}{2}$

$= \frac{27}{2}$ ... that's better.

A note on notation

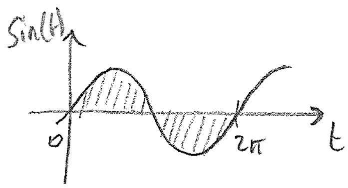
"$F(b) - F(a)$" expression occurs so regularly, it has a shorthand:

$$F(b) - F(a) = F(x)\Big|_a^b = \Big[F(x)\Big]_a^b \qquad \text{all are equivalent.}$$

Eg/.

a) $\displaystyle\int_1^2 x^4\,dx = \frac{1}{5}x^5\Big|_1^2 = \frac{1}{5}(2^5 - 1^5) = \frac{1}{5}(32-1) = \frac{31}{5}.$

b) $\displaystyle\int_0^{2\pi} \sin(t)\,dt = -\cos(t)\Big|_0^{2\pi} = -\left[\cos(2\pi) - \cos(0)\right] = -[1-1] = 0$



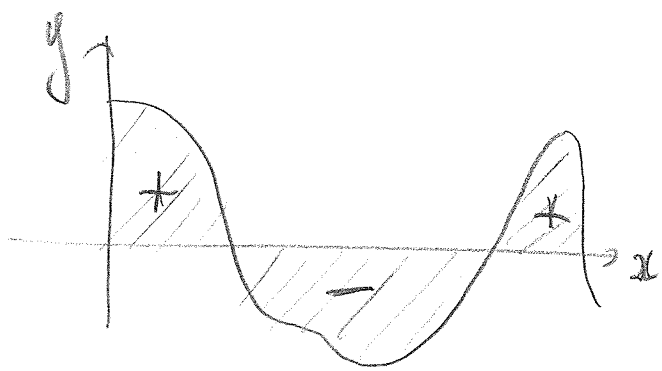
→ Area below x-axis is negative.

Next time:

→ Some important properties of integrals

→ The indefinite integral.

Convention on the 'sign' of an area.



→ Area under the curve is judged relative to x-axis

→ Integral output = Area above − Area below.
