Lecture 28

Today's topics:-

- Second derivative test
- Optimisation.

Read Ch 5.7

Ex 5.7.1 - 5.7.14 (optimisation)

EoL 25.

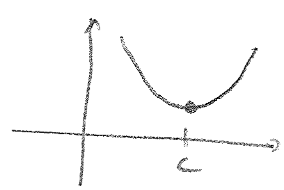
Second Derivative Test

→ Another test for properties of local extrema

→ Use when f'' easy to compute (ow use 1st deriv test)

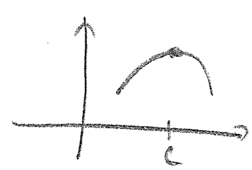
Cases:

a) If $f'(c) = 0$ and $f''(c) > 0$



local min (concave up)

b) If $f'(c) = 0$ and $f''(c) < 0$



local max. (concave down)

c) If $f''(c) = 0$, test is inconclusive.

→ use 1st deriv test. (check f' on left and right)

Examples:

a) $f(x) = x^3 - 9x^2 + 24x.$

$$f'(x) = 3x^2 - 18x + 24$$
$$= 3(x^2 - 6x + 8)$$
$$= 3(x-4)(x-2)$$

critical points : $x = 2, 4.$

$f''(x) = 6x - 18$

$f''(2) = -6 \rightarrow$ local max at $x = 2$

$f''(4) = 6 \rightarrow$ local min at $x = 4.$

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b) $f(x) = x^4.$

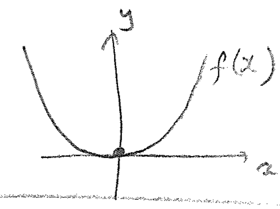
$f'(x) = 4x^3$

critical points $x = 0.$

$f''(x) = 12x^2$

$f''(0) = 0 \Rightarrow$ inconclusive

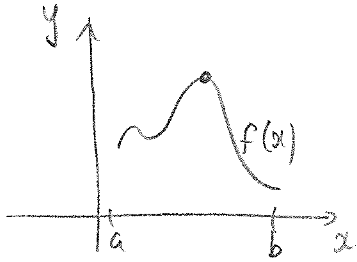
1st deriv test $\Rightarrow$ local min



Optimisation.

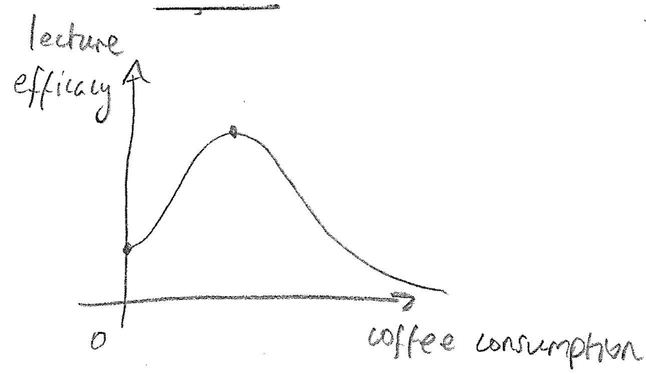
→ Closed interval method disguised in an application.

Before



Find the global max of $f(x)$ on $[a, b]$.

Now

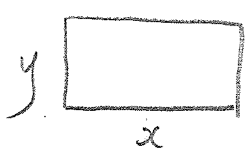


Find coffee consumption that maximises lecture efficacy.

Example 1

Given 100 metres of fence, what is the largest rectangular area that can be contained within the fence? What dimensions does this rectangle have?

Set up.



constraint : $2x + 2y = 100$. (total fence)

maximise area : $A = xy$.

↑ want as a function of a single variable.

→ Use constraint to eliminate a variable.

$$2x + 2y = 100$$

$$\Rightarrow \quad y = \frac{100-2x}{2} = 50 - x$$

domain?

$0 \leq x \leq 50$..

area must be ≥ 0.

Now $\quad A(x) = x(50-x)$

↑

area as a function of x - use CIM to find global max.

$$A(x) = 50x - x^2$$

$$A'(x) = 50 - 2x = 0 \Rightarrow x = 25.$$ ← critical point

Test end points

$$A(0) = A(50) = 0.$$

Test critical point

$$A(25) = 25^2 = 625 m^2. \quad \Rightarrow \text{max area} = 625 m^2$$

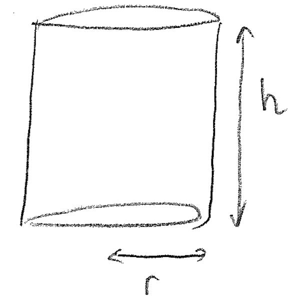
dimensions: $x = 25$

$y = 50 - 25 = 25.$

$25m \times 25m$: a square!

Example 2.

For a cylinder with a fixed surface area S, find the base radius that maximises the volume V.



$V = \pi r^2 h$

$S = \underbrace{2\pi r^2}_{\text{top \& bottom}} + \underbrace{2\pi r h}_{\text{side}}$

→ Write in terms of <u>one</u> variable.

→ Use the fact that S is fixed

$V = \pi r^2 h$

$$S = 2\pi r^2 + 2\pi r h$$

$$\Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r}$$

$$\Rightarrow \quad V(r) = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r} \right) = \frac{1}{2} r (S - 2\pi r^2) \qquad 0 \leq r \leq \sqrt{\frac{S}{2\pi}}$$

<u>Critical pts.</u>

$$V'(r) = \frac{1}{2} S - \pi (3r^2) = 0$$

$$\Rightarrow \quad r^2 = \frac{S}{6\pi} \quad \Rightarrow \quad r = \sqrt{\frac{S}{6\pi}}.$$

Since at end pts $V = 0$ and $V\left(\sqrt{\frac{S}{6\pi}}\right) > 0$,

$r = \sqrt{\frac{S}{6\pi}}$ maximises the volume

Next time: Antiderivatives

→ Reversing the differentiation procedure.

Eg. $f(x) = x^2$

The antiderivative $F(x)$ satisfies $\frac{dF}{dx} = f(x)$.

$F(x) = \frac{1}{3}x^3$ works ....

So does $F(x) = \frac{1}{3}x^3 + \underset{\substack{\uparrow \\ \text{arbitrary constant}}}{C}$

This is the <u>general antiderivative</u>.