~~Le~~cture 21

Today's topics:-

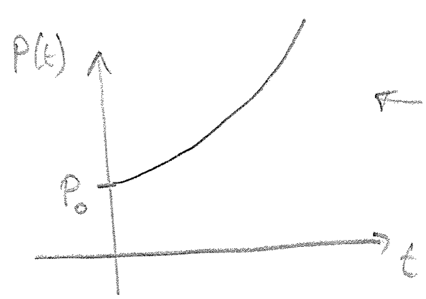
- Interpreting differential equation models
- Exponential processes

- Watch Mobius Lec 19
- EoL 19
- Tutorial today: group work (for credit)

Recall:

$$\underbrace{\frac{dP}{dt} = kP}_{\text{model assumes growth rate prop. to size of P.}}$$

$$\underbrace{P(t) = P_0 e^{kt}}_{\text{has solution of exponential growth } (k>0)}$$



← increases without bound.

More realistic:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{10000}\right)$$

"carrying capacity"

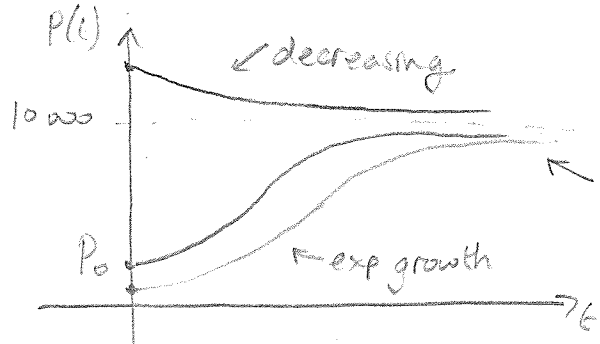
- Solution slightly beyond M127 !!
- How can we interpret such models?

Behaviour of $\frac{dP}{dt}$

For $P \approx 0$, $1 - \frac{P}{10000} \approx 1 \Rightarrow P' \approx kP$ (exp growth)

For $P \approx 10000$, $1 - \frac{P}{10000} \approx 0 \Rightarrow P' \approx 0$ (constant)

For $P > 10000$ $1 - \frac{P}{10000} < 0 \Rightarrow P' < 0$ (decreasing pop)



Further refined model

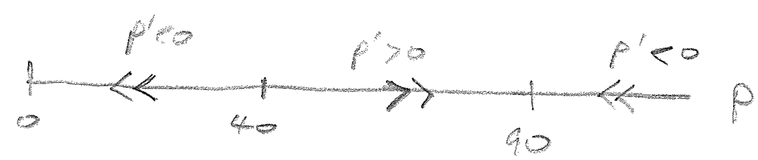
$$\frac{dP}{dt} = kP\left(1 - \frac{40}{P}\right)\left(1 - \frac{P}{90}\right)$$

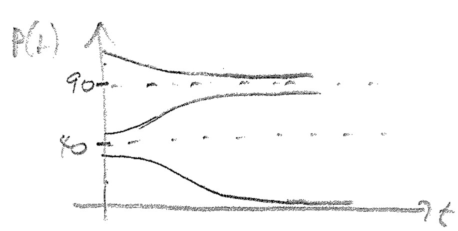
"Allee threshold (40)" due to e.g. difficulty finding a mate for small populations.

"carrying capacity (90)"

When is $\frac{dP}{dt} \gtrless 0$?

Find critical pts where $\frac{dP}{dt} = 0. \Rightarrow P = 0, 40, 90.$





- Population goes extinct for $P < 40$ - the Allee threshold.

Exponential Processes

- Many scientific phenomena involve proportionality between variables & their rates of change

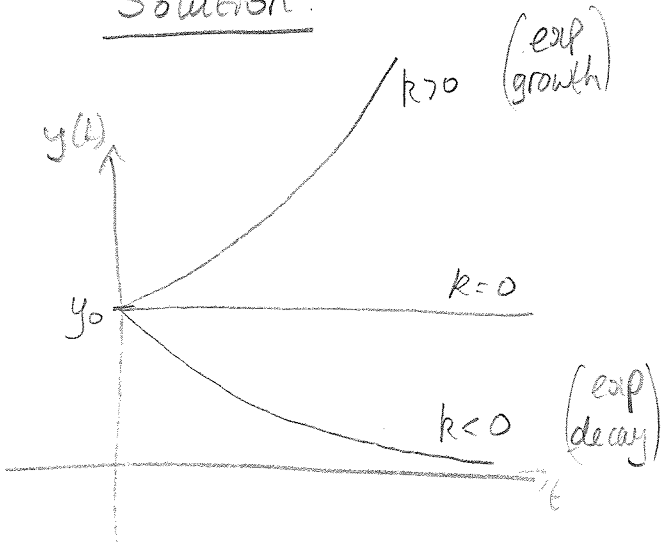
Eg/ pop growth, heat loss, instagram likes ??

$$\frac{dy}{dt} = k y$$

$k > 0$: exponential growth

$k < 0$: exponential decay.

Solution



$$y(t) = y_0 e^{kt}$$

Doubling time ($k > 0$) / Half-life ($k < 0$)

For exponential growth, ($k > 0$) how long before $y(t)$ doubles?

Find t^* such that $y(t^*) = 2y_0$

$$y_0 e^{+kt^*} = 2y_0$$

$$\Rightarrow \quad e^{kt^*} = 2$$

$$\Rightarrow \quad kt^* = \ln 2$$

$$\Rightarrow \quad t^* = \frac{\ln 2}{k}.$$

doubling time

Ex/ Show that half-life (time at which $y(t) = \frac{1}{2} y_0$) for exp. decay is

$$t^* = -\frac{\ln 2}{k}.$$

Example: Radioactive decay.

A particular isotope of californium-250 has a half-life of 13.1 years. 50 mg of the chemical is left on a shelf for 50 years. How many mg remain?

Break it down:

→ exp decay : $y(t) = y_0 e^{kt}$ $(k<0)$.

→ half-life : $y(13.1) = \frac{y_0}{2}$

$\Rightarrow y_0 e^{k(13.1)} = \frac{y_0}{2}$

$\Rightarrow e^{k(13.1)} = \frac{1}{2}$

$\Rightarrow 13.1k = \ln(\frac{1}{2}) \Rightarrow k = \frac{\ln(\frac{1}{2})}{13.1}$.

knowing half-life gives us k!

→ initial mass $y(0) = y_0 = 50\text{mg}$

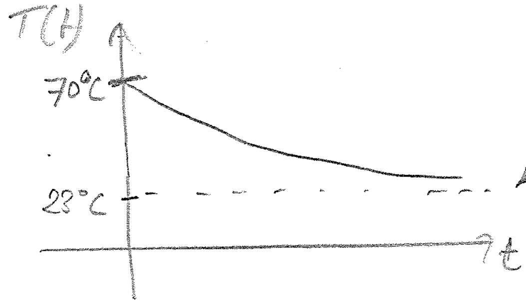
→ predict $y(50)$? $y(50) = 50 e^{\frac{\ln(\frac{1}{2})}{13.1}(50)} \simeq 3.55 \text{ mg}$.

$\therefore$ about 3.55 mg remain after 50 yrs.

Exponential Convergence

We've seen $\begin{cases} \text{exp. growth} & y = e^{kt} \to \infty \text{ as } t \to \infty, \ (k>0) \\ \text{exp decay} & y = e^{kt} \to 0 \text{ as } t \to \infty, \ (k<0) \end{cases}$

What about exponential process of a cooling coffee?



decay to a non-zero state (room temp.)

↳ exponential convergence.