Lecture 29

Today's topics:

Antiderivatives
- general
- specific
- higher-order

Read Ch 6.1 (background to Lec 30).

Ex 6.1.1, 6.1.2
EoL 26

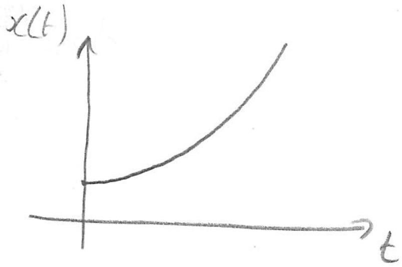
} antiderivs.

Quiz on Monday: - optimisation
- curve sketching

Motivation

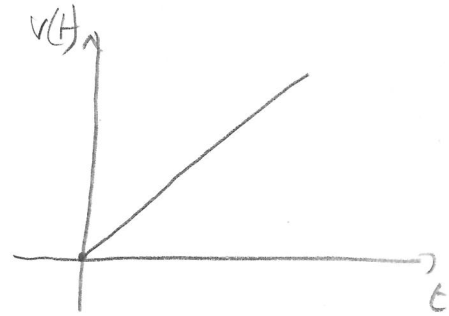
Suppose we are given the position of an object at time t, $x(t)$.

We found the velocity by differentiating: $v = \frac{dx}{dt}$



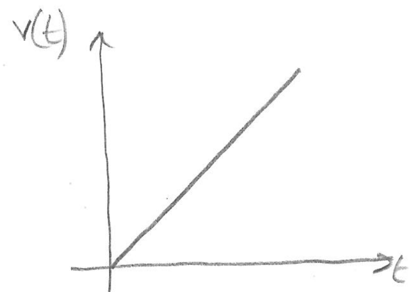
$x(t) = \frac{1}{2}at^2$

$\Rightarrow$



$v(t) = at$

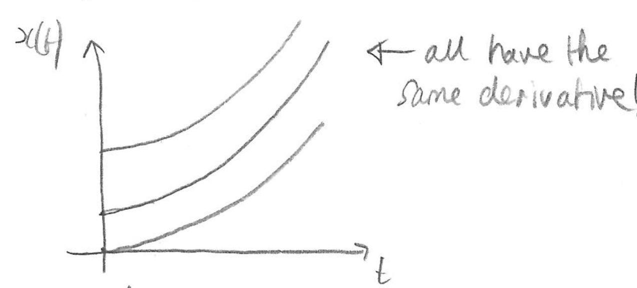
What if we are given $v(t)$ and want to find $x(t)$?



$v(t) = at$

$\Rightarrow$

We say $x(t)$ is an antiderivative of $v(t)$.



There are (infinitely) many of them.

Definition

$F(x)$ is an antiderivative of $f(x)$ if

$$\frac{dF}{dx} = f(x)$$

Example

Find an antiderivative of $f(x) = \cos(x)$

We know $\frac{d}{dx}(\sin(x)) = \cos(x)$.

So $F(x) = \sin(x)$ is an antiderivative.

But so is $F(x) = \sin(x) + 1$.

and $F(x) = \sin(x) + \pi$ hmm...

Fact: All antiderivatives differ only by a constant.

So we define the general antiderative as

$$F(x) + C$$

where $\frac{dF}{dx} = f(x)$ and $C \in \mathbb{R}$ is an arbitrary constant.

Determining Antiderivatives

→ For finding derivatives, we had rules (product, chain etc).

→ For antiderivatives, the strategy is less obvious.

Eg. $\frac{d}{dx}\left(x\ln(x) - x + c\right) = \frac{x}{x} + \ln(x) - 1 = \ln(x)$

$\Rightarrow F(x) = x\ln(x) - x + c$ is the general antiderivative of $\ln(x)$. How could we have known? (later)

→ For now, use differentiation results in reverse.

Examples

a) $f(x) = \frac{1}{1+x^2}$, $\quad F(x) = \arctan(x) + c$

b) $f(x) = \sec^2(x)$ $\quad F(x) = \tan(x) + C$.

c) $f(x) = e^{2x}$. We know $\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$

$\Rightarrow F(x) = \frac{1}{2}e^{2x} + c$

d) $f(x) = 2^x$

We know $\frac{d}{dx}(2^x) = \ln(2)\, 2^x$

$$\Rightarrow \frac{d}{dx}\left(\frac{2^x}{\ln 2}\right) = 2^x$$

$$\Rightarrow F(x) = \frac{2^x}{\ln 2} + C$$

e) $f(x) = x^n$ $(n \neq -1)$.

We know power goes down by 1 upon differentiation. So try

$$\frac{d}{dx}\left(x^{n+1}\right) = (n+1)\,x^n$$

Ok so

$$\frac{d}{dx}\left(\frac{1}{n+1}\,x^{n+1}\right) = x^n$$

$$\Rightarrow F(x) = \frac{1}{n+1}\,x^{n+1} + C$$

see now why $n \neq -1$.

What if $n = -1$?

f) $f(x) = \frac{1}{x}$

$F(x) = \ln(x)$. $(x > 0)$.

↳ can show that $F(x) = \ln(|x|)$ for $x \in \mathbb{R}$.

A Specific Antiderivative.

→ When an additional piece of data is given, we may find c.

Eg. $f'(x) = 1 + 3\sqrt{x}$, $f(4) = 25$, find $f(x)$.

$$f(x) = x + \frac{3x^{(\frac{1}{2}+1)}}{(\frac{1}{2}+1)} + c$$

$$= x + \frac{3x^{3/2}}{(3/2)} + c$$

$$= x + 2x^{3/2} + c.$$

$$f(4) = 4 + 2(4^{3/2}) + c$$
$$= 4 + 2(8) + c$$
$$= 20 + c$$

$f(4) = 25$

$\Rightarrow c = 5.$

$\Rightarrow f(x) = x + 2x^{3/2} + 5$

Higher-order Antiderivatives.

Eg. $f''(x) = e^x - 2\sin(x)$, $f(0) = 3$, $f(\frac{\pi}{2}) = 0$.

$\Rightarrow f'(x) = e^x + 2\cos(x) + C$

$\Rightarrow f(x) = e^x + 2\sin(x) + Cx + D$

↑ new constant.

Ex. find $C = -\frac{2}{\pi}(e^{\frac{\pi}{2}} + 4)$, $D = 2$ from data.