Lecture 24

Today's topics:

Newton's method (2)

L'Hopital's Rule

Read Ch 5.5

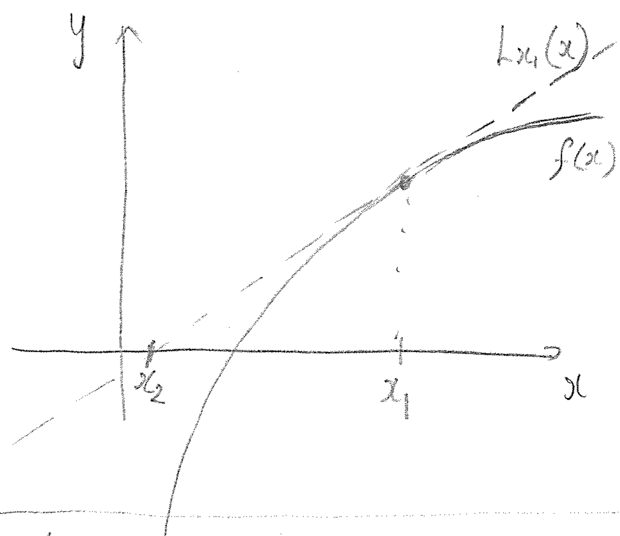
Ex 5.5.1 - 5.5.20

EoL 22

EoL 17-22 and project due Fri

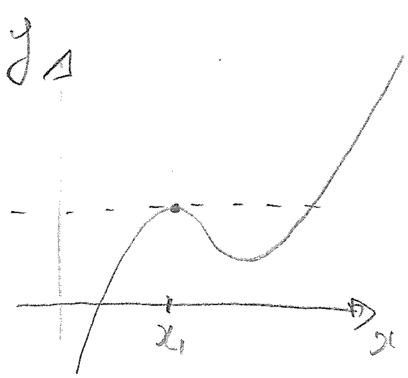
Tutorial today - group work.

Recall: Newton's Method.



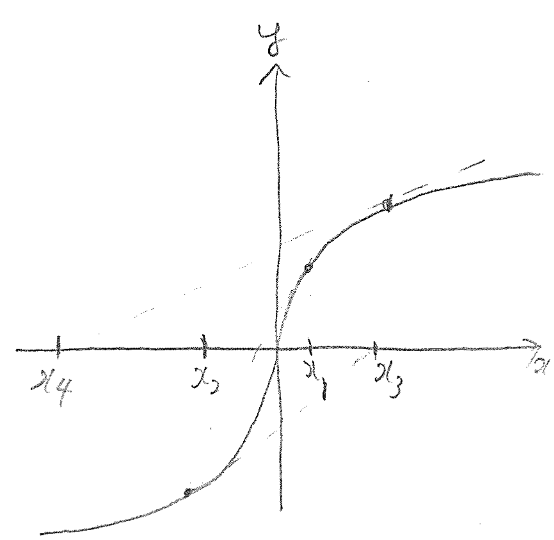
- NM uses tangent line to approximate $f(x)$ and therefore its root.
- What could go wrong?

"NM fails"

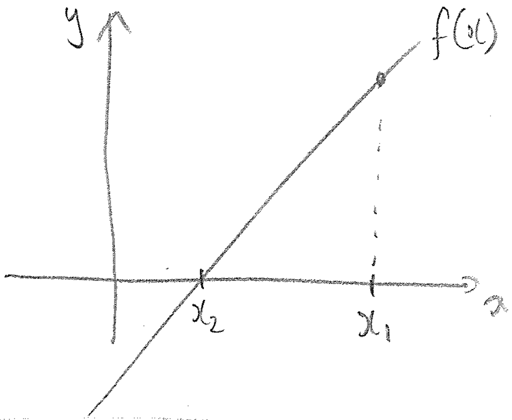


$\leftrightarrow$ $f'(x_1) = 0$
$\Rightarrow$ NM undefined.

$f'(x)$ changes drastically near root
$\Rightarrow$ NM unstable. $\leftrightarrow$



What if $f(x)$ is linear?



- NM works in 1 step! (didn't really need NM).

One more type of root finding problem.

→ Use NM to approximate $\sqrt[8]{500}$. Root finding??

Set $x = \sqrt[8]{500}$ and simplify...

$$x^8 = 500$$
$$\Rightarrow \underbrace{x^8 - 500}_{f(x)} = 0 \qquad \Rightarrow \sqrt[8]{500} \text{ is a root of } f(x).$$

$\therefore$ use NM on $f(x)$ to approximate $\sqrt[8]{500}$.

Initial guess $x_1 = 2$ (since $2^8 = 256$ and 3^8 is way too big).

$f'(x) = 8x^7$.

$$x_2 = 2 - \frac{2^8 - 500}{8(2^7)} \simeq 2.23828\ldots$$

$\vdots$

$x_5 = 2.174559$, vs actual $\sqrt[8]{500} = 2.1745593\ldots$

Back to limits - a reflection.

Some limits could be found immediately:

$$\lim_{x\to 1}(x) = 1, \quad \lim_{x\to\infty}\frac{1}{x^2} = 0, \quad \lim_{x\to 0}\frac{x}{25} = 0$$

Others required more work:

$$\lim_{x\to\infty}\frac{3x^2+1}{2x^2-3} = \text{"}\frac{\infty}{\infty}\text{"}, \quad \lim_{x\to 3}\frac{x^2-9}{x-3} = \text{"}\frac{0}{0}\text{"}.$$

(divide by highest power) (factorize)

Limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are called <u>indeterminate forms</u>.

(says nothing about limit).

Eg/.

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = 6 \quad \left(\text{was } \frac{0}{0}\right).$$

$$\lim_{x\to 0^+}\frac{x}{\sqrt{x}} = 0 \quad \left(\text{was } \frac{0}{0}\right)$$

$$\lim_{x\to 0}\frac{x}{3x^2} = \infty \ (\text{DNE}) \quad \left(\text{was } \frac{0}{0}\right)$$

"$\frac{0}{0}$" could lead to anything!

So far we've resolved indeterminate forms by rewriting them. – not always possible!

Eg/. $\lim_{x\to 0} \frac{\sin(x)}{x}$ $\left(\frac{0}{0}\right)$.

$\lim_{x\to\infty} \frac{\ln(x)}{x}$ $\left(\frac{\infty}{\infty}\right)$

$\Bigg\}$ how can we compute these?

It turns out derivatives can help....

L'Hopital's Rule

If $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$ (or $\pm\infty$) <u>OR</u>

If $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to a$ (or $\pm\infty$), then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

<u>Examples.</u>

a) $\lim_{x\to 0}\frac{\sin(x)}{x}$ $\left(\frac{0}{0}\text{ form}\right)$

$\overset{(H)}{=} \lim_{x\to 0}\frac{\cos(x)}{1} = \frac{\cos(0)}{1} = 1$

b) $\lim_{x\to\infty}\frac{\ln(x)}{x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$\overset{(H)}{=} \lim_{x\to\infty}\frac{(1/x)}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$

$\overset{(H)}{=}$ to indicate that L'Hopital's rule used.

Note: Evaluate derivatives of numerator and denominator separately.
(keep distinct from quotient rule)

Example: exponential vs polynomial

Compute $\lim_{x\to\infty} \frac{x^2}{e^x}$. $\left(\frac{\infty}{\infty}\right)$

$\overset{(4)}{=} \lim_{x\to\infty} \frac{2x}{e^x}$ $\left(\frac{\infty}{\infty}\right)$ - use l'Hopital again!

$= \lim_{x\to\infty} \frac{2}{e^x}$

$= 0$.

How about $\lim_{x\to\infty} \frac{1000x^{1000}}{e^x}$?

$= 0$ by repeated l'Hopital
$\Rightarrow$ exponentials outpace any polynomial as $x \to \infty$.