Lecture 17

Today's topics:

- derivatives of important functions
- product, quotient & chain rule

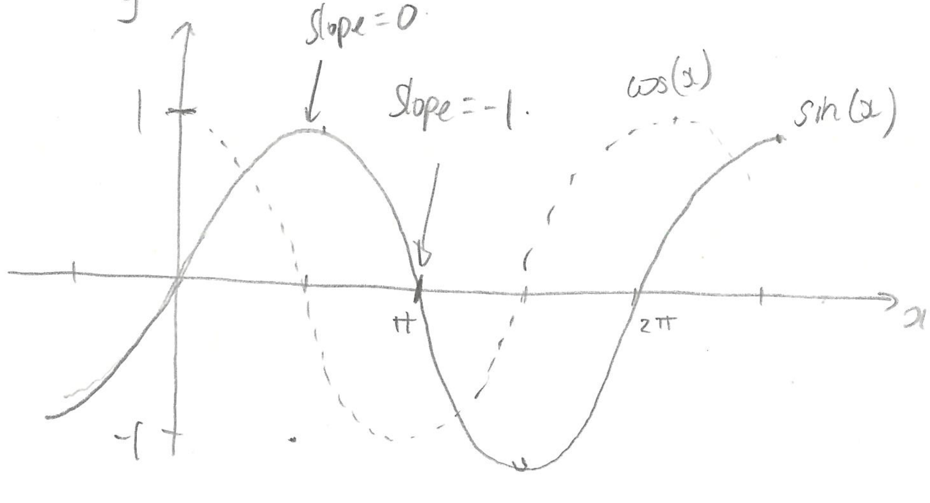
Read Ch 4.5

Ex 4.5 all
4.6 all (good drill qs)

Eoh 15, 16.

Derivatives of important functions

1. $\frac{d}{dx}(b^x) = b^x \ln b \quad (b>0)$
2. $\frac{d}{dx}(e^x) = e^x$ (set $b=e$ in 1.)
3. $\frac{d}{dx}(\sin(x)) = \cos(x)$
4. $\frac{d}{dx}(\cos(x)) = -\sin(x)$
5. $\frac{d}{dx}(\log_b(x)) = \frac{1}{\ln(b)} \frac{1}{x}$
6. $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ (set $b=e$ in 5.)



- slope of $\sin(x)$ is $\cos(x)$!
- slope of e^x is itself!

The product rule

Derivative of $f(x)g(x)$?

Maybe $f'(x)g'(x)$? No. Eg/ $\underbrace{\frac{d}{dx}(1.x)}_{1} \neq \underbrace{\frac{d}{dx}(1)\frac{d}{dx}(x)}_{0}$

Given by the product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Example:

$$g(x) = (x + 2\sqrt{x})e^x$$

$$g'(x) = \left(1 + 2\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)\right)e^x + (x + 2\sqrt{x})e^x$$

$$= \left(1 + \frac{1}{\sqrt{x}} + x + 2\sqrt{x}\right)e^x.$$

The Quotient Rule.

ordering of derivatives matter here.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

(cheat sheet!)

$$= \text{"}\frac{(bot)(top') - (top)(bot')}{bot^2}\text{"}.$$

Example:

$$v(t) = \frac{4+t}{te^t}$$

$$v'(t) = \frac{te^t(4+t)' - (4+t)(te^t)'}{(te^t)^2}$$

$$= \frac{te^t - (4+t)(e^t + te^t)}{t^2e^{2t}}$$ product rule

$$= \frac{-4-4t-t^2}{t^2e^t}$$ simplifying steps optional.

Example:

Prove $\frac{d}{dx}(\tan(x)) = \sec^2(x)$

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)(\sin(x))' - \sin(x)(\cos(x))'}{\cos^2(x)}$$

$$= \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}$$ $\cos^2 x + \sin^2 x = 1$.

$$= \frac{1}{\cos^2(x)}$$

$$= \sec^2(x)$$ $= \left(\frac{1}{\cos(x)}\right)^2$

The Chain Rule

Derivative of the composition of functions:

$$\frac{d}{dx}\left[f(g(x))\right] = \underbrace{f'(g(x))}_{\text{"outside derivative" evaluated at } g(x)} \underbrace{g'(x)}_{\text{"inside" derivative}}$$

Example

Differentiate $f(x) = (1+x+x^2)^{99}$

Let $g(x) = x^{99}$, $h(x) = 1+x+x^2$ } easy to differentiate.

"outside fn" "inside fn"

$g'(x) = 99x^{98}$

$h'(x) = 1+2x$ } helper steps optional

$f(x) = g(h(x))$

$\Rightarrow f'(x) = g'(h(x))\, h'(x)$

$= 99(1+x+x^2)^{98}(1+2x)$

Example

$f(z) = e^{\frac{z}{z-1}}$ $\left[g(z) = e^z,\ h(z) = \frac{z}{z-1}\right]$

$f'(z) = e^{\frac{z}{z-1}} \cdot \left(\frac{z}{z-1}\right)'$

$= e^{\frac{z}{z-1}}\left(\frac{(z-1)(1) - z(1)}{(z-1)^2}\right)$

$= e^{\frac{z}{z-1}}\left(\frac{-1}{(z-1)^2}\right)$

Avoiding the Quotient Rule.

Find derivative of $f(x) = \dfrac{e^x}{3x+4}$.

Write as $f(x) = e^x(3x+4)^{-1}$

$$\Rightarrow f'(x) = e^x\left[(3x+4)^{-1}\right]' + (e^x)'(3x+4)^{-1}$$

$$= e^x\left[-(3x+4)^{-2}(3)\right] + e^x(3x+4)^{-1}$$

$$= e^x\left(\frac{-3+3x+4}{(3x+4)^2}\right)$$ (simplification optional)

$e^x(3x+4)^{-2}\left[(3x+4)' - 3\right]$