Lecture 13

→ EoL 12.

→ Example 3.40

→ Ex. 3.6.4

→ Survey

→ EoL 8-12 due Fri Oct 12.

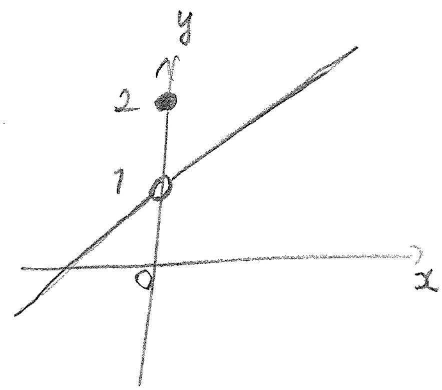
Today's topics:

- limits at discontinuities
- squeeze theorem.

Limit at a point : example.

$$f(x) = \begin{cases} x+1 & x \neq 0 \\ 2 & x = 0 \end{cases}$$

"piecewise defined function".

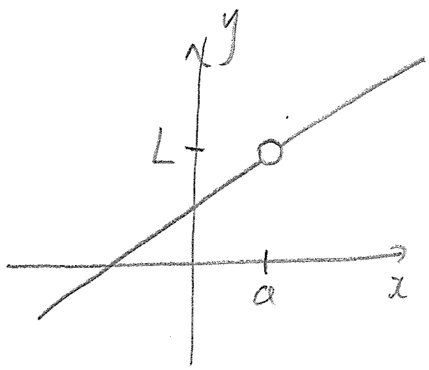


Since $f(0)=2$ does $\lim_{x \to 0} f(x) = 2$? NO!

In computing limit can ignore behaviour <u>at</u> point

As $x \to 0$ we have $f(x) \to 1$. $\Rightarrow \lim_{x \to 0} f(x) = 1$.

Types of discontinuities.

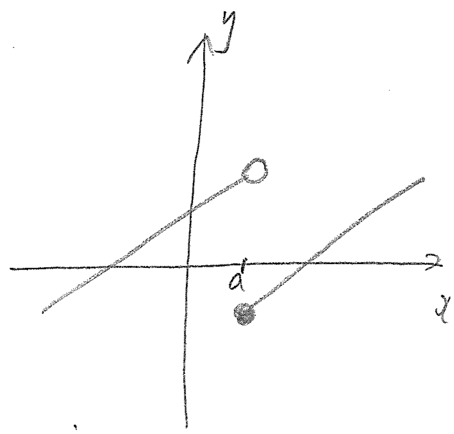


removable discontinuity

$$\lim_{x \to a} f(x) = L$$

$f(a)$ not defined

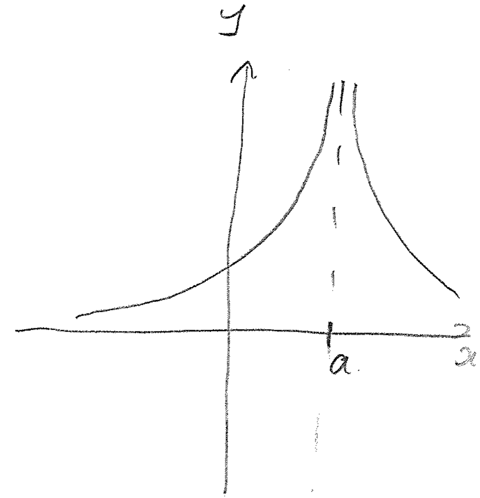
OR $f(a) \neq L$.



jump discontinuity

$$\lim_{x \to a} f(x) = ?$$

$f(a)$ defined



infinite discontinuity

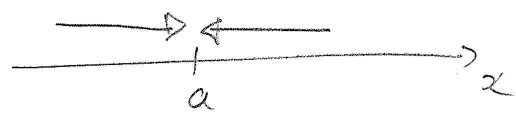
$$\lim_{x \to a} f(x) = \pm\infty$$

$f(a)$ undefined.

Technical point

$$\lim_{x \to a} f(x)$$

"x approaches a" is interpreted as approaching from left and right.

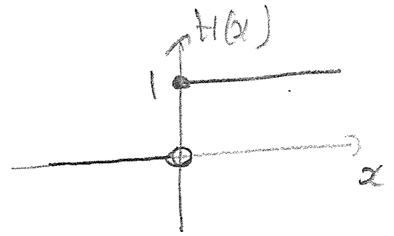


Why care?

Can get different limits:

Eg/ $H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

"Heaviside" function.



$\lim_{x \to 0} H(x)$?   From left, get 0.
From right, get 1.

More specific limits

Limit from the left $(x < a)$: $\lim_{x \to a^-} f(x) = L_1$

Limit from the right $(x > a)$: $\lim_{x \to a^+} f(x) = L_2$

Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

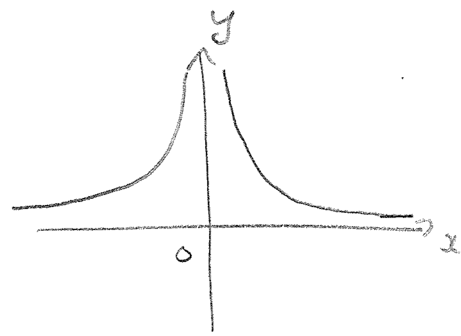
Otherwise, we say limit <u>does not exist</u>.

$\Rightarrow$ $\lim_{x \to 0} H(x)$ does not exist.

But $\lim_{x \to 0^-} H(x) = 0$, $\lim_{x \to 0^+} H(x) = 1$.

The infinite discontinuity

Eg/ $f(x) = \frac{1}{x^2}$



As $x \to 0$, $f(x) \to \infty$.

OR $\lim_{x \to 0} f(x) = \infty$.

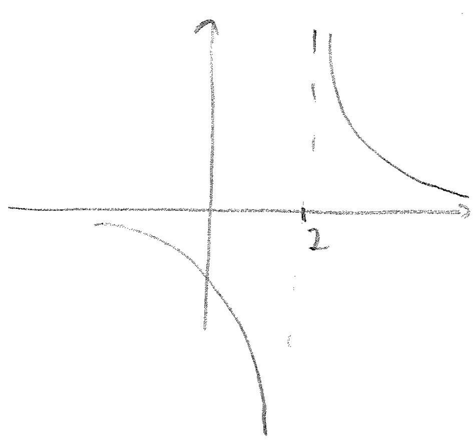
However, technically the limit does not exist. (L must be a number).

Asymptotes.

- Infinite discontinuities $\longleftrightarrow$ Vertical asymptotes
- Limits as $x \to \pm\infty$ $\longleftrightarrow$ Horizontal asymptotes.

Eg/

$y = \frac{1}{x-2}$



$\lim_{x \to 2^+} \frac{1}{x-2} = \text{"}\frac{1}{\text{small +ve}}\text{"} = \infty$

$\lim_{x \to 2^-} \frac{1}{x-2} = \text{"}\frac{1}{\text{small -ve}}\text{"} = -\infty$

Vertical asymptote at $x = 2$.

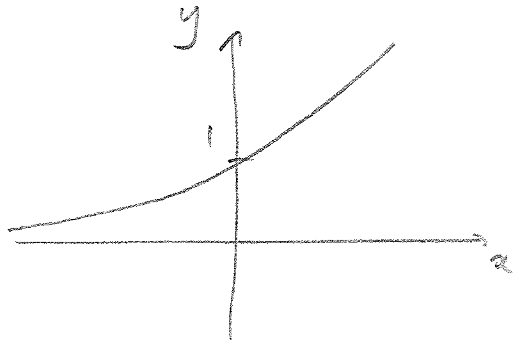
$\lim_{x \to \infty} \frac{1}{x-2} = 0$

$\lim_{x \to -\infty} \frac{1}{x-2} = 0$

H.A at $y = 0$

Eg/

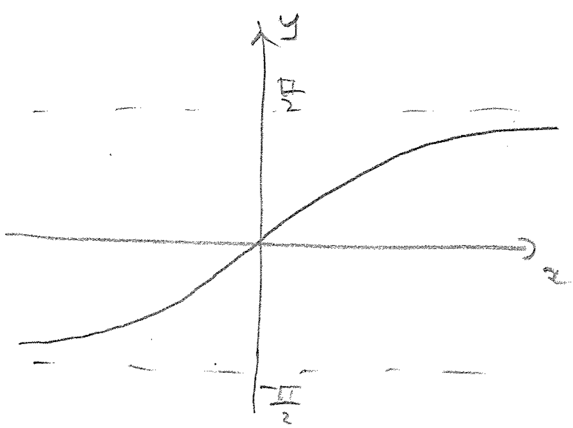
$y = e^x$.



$\lim_{x \to \infty} e^x = \infty$.

$\lim_{x \to -\infty} e^x = 0 \Rightarrow$ H.A at $y = 0$.

$y = \tan^{-1}(x)$



$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$

$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$

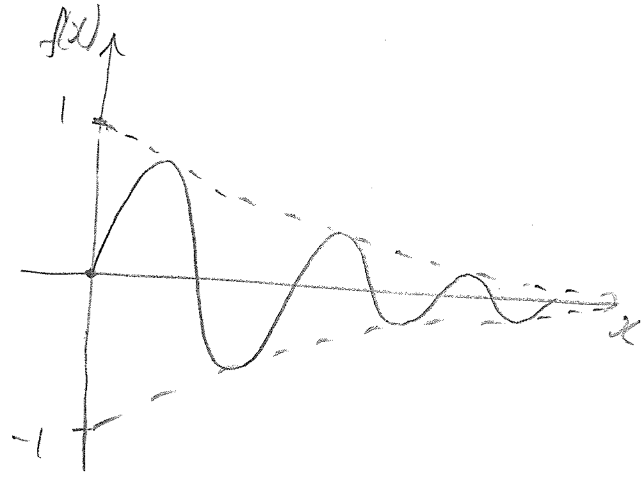
$\Rightarrow$ H.A at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$

The Squeeze Theorem

- Use when there is a bound on a function
  Eg trig. $-1 \leq \sin x \leq 1$, $-1 \leq \cos x \leq 1$.

Eg/ $f(x) = e^{-x} \sin x$

Find $\lim_{x \to \infty} e^{-x} \sin x$.



$-1 \leq \sin x \leq 1$

$-e^{-x} \leq e^{-x} \sin x \leq e^{-x}$

We have

$$-e^{-x} \leq e^{-x}\sin x \leq e^{-x} \qquad (f(x) \text{ is sandwiched})$$

As $x \to \infty$, $e^{-x} \to 0$

$-e^{-x} \to 0$.

$(f(x)$ is squeezed!$)$

$\Rightarrow$ $f(x) \to 0$ by the squeeze theorem

ie. $\lim_{x \to \infty} e^{-x}\sin x = 0$.

Formal squeeze theorem

If $f(x) \leq g(x) \leq h(x)$ for x near a

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$,

then $\lim_{x \to a} g(x) = L$

Eg2/ $f(x) = x^2 \sin\left(\frac{1}{x}\right)$

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$? $\sin\left(\frac{1}{x}\right)$ goes crazy near $x=0$ - but that's ok.

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\Rightarrow \quad \underset{\substack{\downarrow \\ 0}}{-x^2} \leq x^2 \sin\left(\frac{1}{x}\right) \leq \underset{\substack{\downarrow \\ 0}}{x^2} \quad \text{as } x \to 0$$

$\Rightarrow$ $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ by Squeeze Thm.