Lecture 16.

Today's topics:
- The derivative as a function.
- Differentiability (is f differentiable?)
- Differentiation rules.

Read Ch 4.3
Ex. 4.3.1 - 4.3.4.
EoL 14

Previously...

→ Derivative at a point $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

→ Consider a now not as a point, but as an input to the function f'.

Definition: The derivative as a function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is the derivative function of $f(x)$, and gives the slope of the tangent line at $(x, f(x))$.

The domain of f' is the set of x-values for which the limit exists.

Example:

Let $f(x)=\sqrt{x}$. Find $f'(x)$ using defn of derivative and state its domain.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

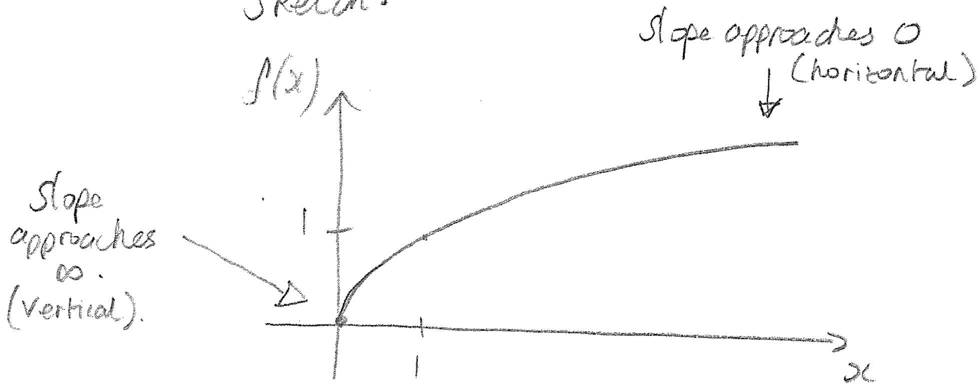
$$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Domain of $f'(x)$ is $(0,\infty)$. ← note $D_{f'}$ does not necessarily equal D_f.

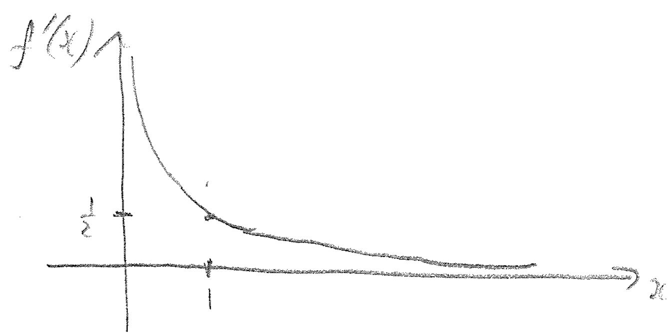
f' tells us the slope of $f(x)$ at any point in $D_{f'}$.

Sketch:



$$\lim_{x\to 0} f'(x) = \infty.$$

$$\lim_{x\to \infty} f'(x) = 0.$$



Definition: 'differentiable'.

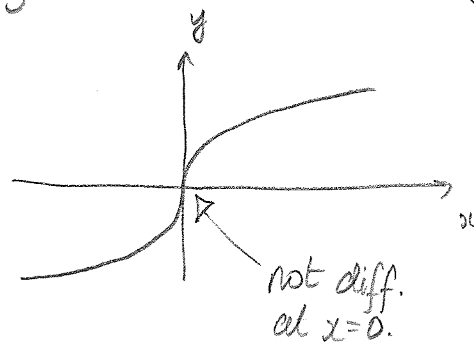
A function f is differentiable at $x=a$ if $f'(a)$ exists.

A function is differentiable on (a,b) if $f'(x)$ exists for every $x \in (a,b)$

Failing to be differentiable.

$$\left(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ must exist!} \right).$$

① Vertical tangent lines.

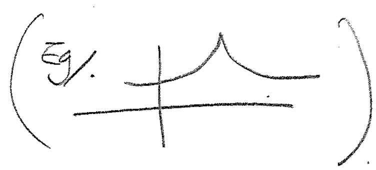


$f(x) = x^{\frac{1}{3}}$

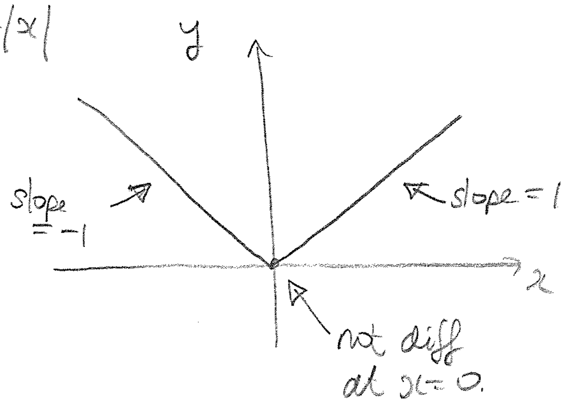
$f'(x) = \dfrac{1}{3x^{2/3}}$.

---

② Corners / cusps.



Eg. $f(x)=|x|$



$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{|h|-0}{h}$$

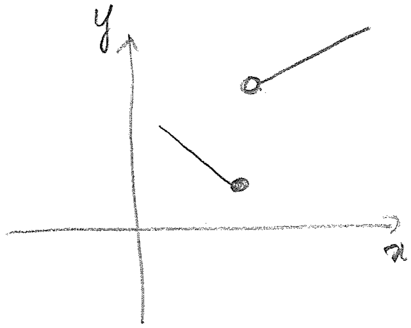
$$= \lim_{h\to 0^+} \frac{h}{h} = 1.$$

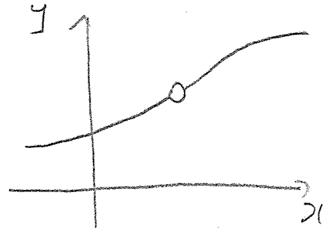
limit does not exist

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

(3) Discontinuities

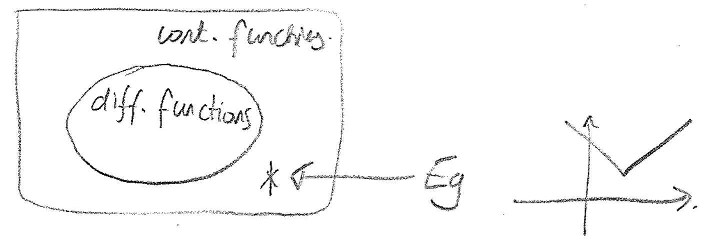




differentiability $\Rightarrow$ continuity

"all differentiable functions are continuous".

Venn. Diag.



Notation.

Equivalent symbols for the derivative

$$f'(x) = y' = \frac{dy}{dx} = \underbrace{\frac{d}{dx}}\left(f(x)\right) = \underbrace{D}f(x)$$

operators : I intend to differentiate $(\cdot)$

Higher-order derivatives

$y = f(x)$ has derivative

$\frac{d}{dx}(f(x)) = f'(x) = \frac{dy}{dx}$: has derivative

$\frac{d}{dx}(f'(x)) = f''(x) = \frac{d^2y}{dx^2}$

$f^{(n)}(x) = \frac{d^ny}{dx^n}$ - "nth derivative of $f(x)$"

Differentiation Rules (way faster!)

→ want proofs? see Ch 4.3!

Power rule: $\frac{d}{dx}(x^a) = a x^{a-1}$ for all $a \in \mathbb{R}$

Constant multiple rule: $\frac{d}{dx}(c f(x)) = c \frac{d}{dx}(f(x))$. for all $c \in \mathbb{R}$

$$\frac{d}{dx}(c) = \frac{d}{dx}(c x^0) = c\frac{d}{dx}(x^0) = 0 .$$

constant multiple out

power rule

The derivative of a constant is zero!

Adding functions: $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

Example

$$f(x) = 4x^{5/3} - \frac{2}{x} + \pi$$

$$f'(x) = \frac{d}{dx}(4x^{5/3}) - \frac{d}{dx}\left(\frac{2}{x}\right) + \frac{d}{dx}(\pi)$$

$$= 4\frac{d}{dx}(x^{5/3}) - 2\frac{d}{dx}(x^{-1}) + \pi\frac{d}{dx}(1)$$

$$= 4 \cdot \frac{5}{3}x^{2/3} - 2(-x^{-2}) + 0$$

$$= \frac{20}{3}x^{2/3} + \frac{2}{x^2}.$$

Derivatives of Transcendental Functions

1. $\frac{d}{dx}(e^x) = e^x$

2. $\frac{d}{dx}(b^x) = b^x \ln(b) \qquad (b>0)$

3. $\frac{d}{dx}(\sin(x)) = \cos(x)$

4. $\frac{d}{dx}(\cos(x)) = -\sin(x)$

5. $\frac{d}{dx}(\log_b(x)) = \frac{1}{\ln(b)} \frac{1}{x}$.

Ch 4.4, 4.6.

(can prove using def^n of derivative)