Lecture 18

Today's topics:

- Nested chain rule.
- Finding tangent lines
- Implicit differentiation.

Read Ch 4.7 - 4.8

Ex 4.7.1, 4.8.6, 4.8.7, 4.9.5, 4.9.6 (a-c)

EoL 17.

Warm up: Chain rule practise.

a).

$$f(x) = (x^2+1)^3$$

$$f'(x) = 3(x^2+1)^2 . (2x)$$

$g(x) = x^3$

$h(x) = x^2+1$

$f(x) = g(h(x))$

$f'(x) = g'(h(x))\, h'(x).$

b).

$$f(x) = a^{(b^x)}.$$

Recall $\frac{d}{dx}(a^x) = a^x \ln(a)$

$$f'(x) = \underbrace{a^{b^x} \ln(a)}_{\text{outside deriv}} \times \underbrace{b^x \ln(b)}_{\text{inside deriv.}}$$

$g(x) = a^x$

$h(x) = b^x$

$f(x) = g(h(x))$

$f'(x) = g'(h(x))\, h'(x)$

Nested Chain Rule.

nested functions

$$\frac{d}{dx}\left[f(g(h(x))) \right] = f'[g(h(x))]\, g'(h(x))\, h'(x).$$

Eg

$$\alpha(x) = \sin((x^2+1)3)$$

$$\alpha'(x) = \cos((x^2+1)^2) . 3(x^2+1)^2 . 2x.$$

$f(x) = \sin x$

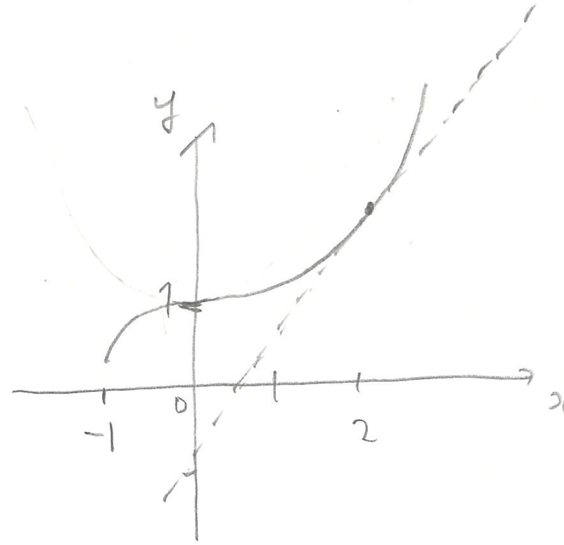
$g(x) = x^3$

$h(x) = x^2+1$

Finding tangent lines

Find the equation for the tangent line to $y = \sqrt{1+x^3}$ at the point $(2,3)$.

not just derivative.



Step 1: Find slope of tangent line at $x=2$.

$$y' = \frac{1}{2}(1+x^3)^{-\frac{1}{2}}(3x^2) = \frac{3x^2}{2\sqrt{1+x^3}}$$

$$y'(2) = \frac{3(4)}{2\sqrt{9}} = 2$$

Step 2: Find equation of line:

- gradient $m = 2$.
- passes through $(x_1, y_1) = (2,3)$

$$y - y_1 = m(x - x_1)$$

(OR $y = mx + c$ if you prefer)

$$\Rightarrow \quad y - 3 = 2(x-2)$$

$$\Rightarrow \quad y = 2x - 1.$$ ← eqn of tangent line to $y = \sqrt{1+x^3}$ at $(2,3)$.

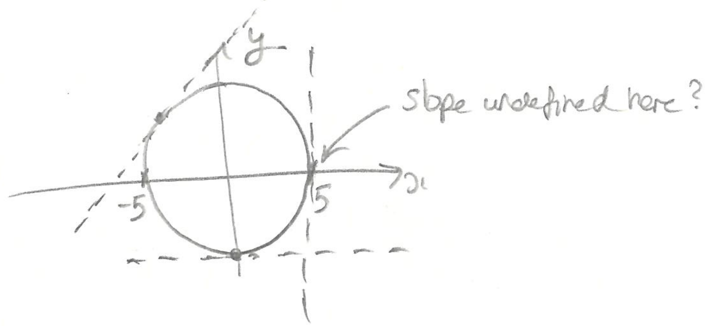
Implicit Differentiation - Motivation

→ So far we've seen $y = f(x)$: find $y' = f'(x)$.

function - passes VLT.

→ What about equations that are not functions?

Eg/ $x^2 + y^2 = 25$



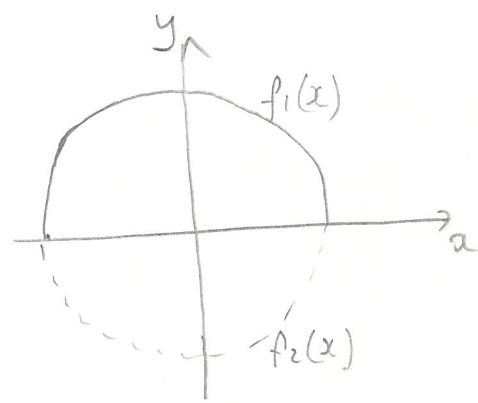
→ Can we find slope of these equations?

Method 1: Express equation as a set of functions: $y = f(x)$

Eg/ $x^2 + y^2 = 25$.

$\Rightarrow y = \pm\sqrt{25 - x^2}$.

$f_1(x) = \sqrt{25 - x^2}$
$f_2(x) = -\sqrt{25 - x^2}$ } both satisfy VLT.



Compute derivative as usual:

$$f_1'(x) = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{25 - x^2}}.$$

$$f_2'(x) = \frac{x}{\sqrt{25 - x^2}}.$$

Sometimes difficult (or impossible) to isolate y!

Eg. $xy^2 + e^y = 0$.

Method 2: Use Implicit Differentiation

Key idea: treat y as a function of x and use the chain rule.

"$y = f(x)$"

Eg.
$$\frac{d}{dx}(y^2) = \underbrace{\frac{d}{dx}\left([f(x)]^2\right) = \overbrace{2f(x)\frac{d}{dx}f(x)}^{\text{chain rule}}}_{\text{normally omit these steps.}} = 2y\frac{dy}{dx}.$$

Eg.
$$\frac{d}{dx}(\sin y) = (\cos y)\frac{dy}{dx}$$

Eg.
$$\begin{aligned}\frac{d}{dx}(xy^2) &= x'y^2 + x(y^2)' && \text{(product rule)}\\ &= 1.y^2 + x\left[2y\frac{dy}{dx}\right] && \text{(chain rule)}\\ &= y^2 + 2xy\frac{dy}{dx}.\end{aligned}$$

Back to circle...

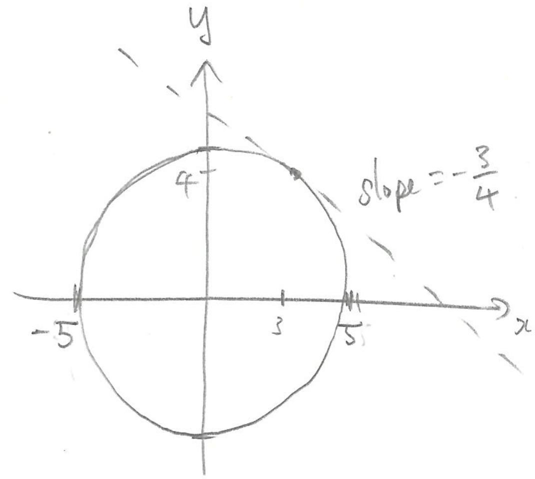
$$x^2 + y^2 = 25$$

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$



Slope at point $(3, 4)$?

$$\frac{dy}{dx} = -\frac{3}{4}$$

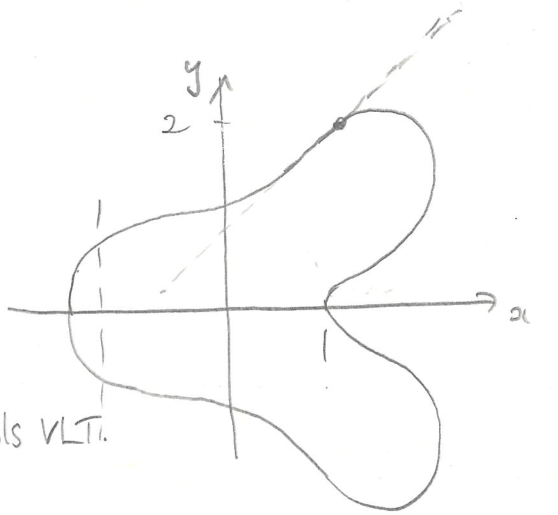
Note: We didn't need to pick a function (as required for method 1).

Example:

Determine slope of tangent line to curve

$$x^4 - 4xy^2 + y^4 = 1$$ ← tricky to isolate y.

at point $(1, 2)$.



fails VLT.

$$\frac{d}{dx}\left(x^4 - 4x\,y^2 + y^4\right) = \frac{d}{dx}(1)$$

$$4x^3 - 4y^2 - 4x(2yy') + 4y^3y' = 0$$

Can solve for y'.

Faster here to just sub in point $x = 1$, $y = 2$.

$$4(1) - 4(4) - 4(1)(2(2)y') + 4(8)y' = 0$$

$$\Rightarrow \quad 4 - 16 - 16y' + 32y' = 0$$

$$\Rightarrow \quad 16y' = 12$$

$$\Rightarrow \quad y' = \frac{3}{4}.$$