Lecture 35

Today's topics:

Integration by parts

Read Ch 7.4

Ex 7.4.1 - 7.4.11

EoL 33 (last one!)

Project 3 due 11.59pm Mon

EoLs & Bi-weekly 6 - 4pm Mon

Review sessions - poll

Recall: The product rule

The derivative of the product of two functions:

$$\frac{d}{dx}\left(u(x)v(x)\right) = u'(x)v(x) + u(x)v'(x)$$

Now integrate both sides

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

Integration 'reversed' the $\frac{d}{dx}$.

Now rearrange:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx.$$

formula for integration by parts (IBP).

Another form:

note that $\frac{du}{dx} = u'(x) \Rightarrow du = u'(x)\,dx$

Likewise $dv = v'(x)\,dx$.

$$\Rightarrow \int u\,dv = uv - \int v\,du$$

How is this useful?

$$\underbrace{\int u(x)\,v'(x)\,dx}_{\text{original integral - tricky}} = u(x)v(x) - \underbrace{\int v(x)\,u'(x)\,dx}_{\text{new integral - hopefully easier}}.$$

Eg

$$\int \underbrace{x}_{u(x)}\,\underbrace{e^x}_{v'(x)}\,dx = \underbrace{x}_{u(x)}\,\underbrace{e^x}_{v(x)} - \underbrace{\int \underbrace{e^x}_{v(x)}\cdot\underbrace{1}_{u'(x)}\,dx}_{\text{easier!}}$$

Pick these.

$$= xe^x - e^x + C.$$

Observations on choice of $u(x)$, $v'(x)$.

→ must be able to find antiderivative of $v'(x)$

→ choose $u(x)$ such that $\int v u'\,dx$ is 'nice'.

Let's try the other choice of $u(x)$ and $v'(x)$

$$\int \underset{v'(x)}{\underset{\uparrow}{x}}\ \underset{u(x)}{\underset{\uparrow}{e^x}}\, dx = \underset{u(x)}{\underset{\uparrow}{e^x}} \underset{v(x)}{\underset{\uparrow}{\left(\tfrac{1}{2}x^2\right)}} - \int \underset{v(x)}{\underset{\uparrow}{\tfrac{1}{2}x^2}}\ \underset{u'(x)}{\underset{\uparrow}{e^x}}\, dx$$

$$= \frac{1}{2}x^2 e^x - \frac{1}{2}\underbrace{\int x^2 e^x \, dx}_{\text{tricky!}}.$$

IBP made our integral worse!
(Try to pick $u(x)$ so that $u'(x)$ simplifies the integral

Eg/. $\displaystyle\int x^2 \sin x \, dx$

IBP may need to be applied twice.

Pick $u(x) = x^2$ $\Rightarrow$ $u'(x) = 2x$
$v'(x) = \sin x$ $\qquad$ $v(x) = -\cos x$

$$\Rightarrow \int x^2 \sin x \, dx = -x^2\cos x - \int -\cos x \cdot 2x \, dx.$$

$$= -x^2 \cos x + 2\underbrace{\int x \cos x \, dx}_{\text{IBP again!}}$$

$$= -x^2 \cos x + 2I$$

$$I = \int x \cos x \, dx = x\sin x - \int \sin x \, dx$$

$\left[\begin{array}{l} u = x \\ v' = \cos x \end{array}\right]$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

Sub I into main integral:

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2(x \sin x + \cos x + c)$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + c_2$$

New constant $(=2c)$

Check ans:

$$\frac{d}{dx}\left[-x^2 \cos x + 2x \sin x + 2\cos x + c_2\right]$$

$$= -\cancel{2x \cos x} + x^2 \sin x + \cancel{2\sin x} + \cancel{2x\cos x} - \cancel{2\sin x}$$

$$= x^2 \sin x \quad \checkmark$$

A clever use of IBP.

$$\int \ln x \, dx = \int \underbrace{1}_{v'(x)} . \underbrace{\ln(x)}_{u(x)} \, dx.$$

$v(x) = x$

$u'(x) = \frac{1}{x}$.

$$= x \ln x - \int x\left(\frac{1}{x}\right) dx.$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

IBP for definite integrals

Note
$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx$$

Eg/. $\displaystyle\int_1^e x\ln x\,dx.$

Pick $u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x}$
$v'(x) = x \Rightarrow v(x) = \frac{1}{2}x^2$

$$\Rightarrow \int_1^e x\ln x\,dx = \frac{1}{2}x^2\ln(x)\Big|_1^e - \int_1^e \frac{1}{2}x^2\left(\frac{1}{x}\right)dx$$
$$= \frac{1}{2}e^2\ln(e) - \frac{1}{2}\ln(1) - \frac{1}{2}\int_1^e x\,dx$$
$$= \frac{1}{2}e^2 - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^e$$
$$= \frac{1}{2}e^2 - \frac{1}{4}(e^2-1).$$
$$= \frac{1}{4}(e^2+1).$$