Lecture 36.

Today's topics:

- 3 chunky integration examples.

- Cheat sheet recommendations.
- Complete doodle poll for exam review sessions.
- Please evaluate the course. https://evaluate.uwaterloo.ca.
  - closes tonight.
- Tutorial - project help.

Recall: Integration by Parts.

$$\int u v' \, dx = uv - \int v u' \, dx$$

In choosing u, can use the rule of thumb: (LIATE)

	Eg.
Logs	$\ln(x)$
Inverse trig	$\sin^{-1}(x)$
Algebraic exp.	x^2
Trig	$\cos(x)$
Exponentials.	e^x

Whichever comes first, assign as u.

Eg. $\int x^2 e^x \, dx$
$u = x^2$ $v' = e^x$.

$\int x \ln(x)$
$u = \ln(x)$
$v' = x$

$\int \sin^{-1}(x) \, dx$
$u = \sin^{-1}(x)$
$v' = 1$

/

Example. (IBP and substitution)

Qn/ Use IBP followed by a substitution to find

$$\int \tan^{-1}(x)\,dx.$$

Ans/ $u(x) = ?$

$v'(x) = ?$

We don't know the antideriv of $\tan^{-1}(x)$, So must use $u(x) = \tan^{-1}(x)$ and $v'(x) = 1$.

$$\Rightarrow \quad u'(x) = \frac{1}{1+x^2}$$

$$v(x) = x.$$

Sub into formula for IBP:

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \underbrace{\int \frac{x}{1+x^2}\,dx}_{\text{compute using a substitution.}}$$

$$\int \frac{x}{1+x^2}\,dx$$

$u = 1 + x^2$

$$\Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx.$$

$$\Rightarrow \int \frac{x}{1+x^2}\,dx = \int \frac{1}{u}\left(\frac{du}{2}\right)$$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$\Rightarrow \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln\left|1+x^2\right| + c$$

Put it all together:

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \left(\frac{1}{2}\ln\left|1+x^2\right| + c\right)$$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln\left|1+x^2\right| + c_2$$

Check answer by differentiating

$$\frac{d}{dx}\left[x\tan^{-1}(x) - \frac{1}{2}\ln\left|1+x^2\right| + c_2\right]$$

$$= x \times \frac{1}{1+x^2} + 1.\tan^{-1}(x) - \frac{1}{2}\frac{1}{1+x^2}(2x) + 0.$$

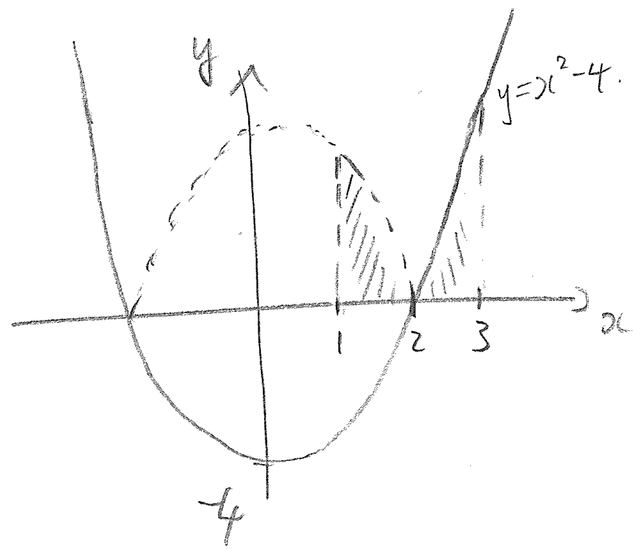
$$= \frac{x}{1+x^2} + \tan^{-1}(x) - \frac{x}{1+x^2}$$

$$= \tan^{-1}(x). \checkmark$$

Example (Integrating absolute values)

Qn. $\int_1^3 |x^2-4|\,dx$

Sketch:



Algebraically

$$|x^2-4| = \begin{cases} -(x^2-4) & 1 \le x \le 2 \\ x^2-4 & 2 \le x \le 3 \end{cases}$$

Split up integral

$$\int_1^3 |x^2-4|\,dx = \int_1^2 |x^2-4|\,dx + \int_2^3 |x^2-4|\,dx$$

$$= \int_1^2 -(x^2-4)\,dx + \int_2^3 (x^2-4)\,dx$$

$$= -\left[\tfrac{1}{3}x^3 - 4x\right]_1^2 + \left[\tfrac{1}{3}x^3 - 4x\right]_2^3$$

$$= -\left(\tfrac{8}{3} - 8 - \left(\tfrac{1}{3}-4\right)\right) + \left(\tfrac{27}{3} - 12 - \left(\tfrac{8}{3}-8\right)\right)$$

$$= 4.$$

Example (The FTOC)

FTOC

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$

(F is antiderivative of f).

Qn/ The temperature of the Earth, T, as a function of distance from the centre, r, satisfies

$$\frac{dT}{dr} = -\frac{75}{2}\frac{1}{\sqrt{6400-r}}.$$

Given that temp. at $r = 6400$km is $0°C$, find temp. at Earth's core $(r=0)$.

Ans/

Given: $T(6400) = 0$.

Find: $T(0)$

FTOC:

$$\underbrace{\int_{r=0}^{r=6400} \frac{dT}{dr}\,dr}_{\text{can compute}} = \underbrace{T(6400)}_{\text{know}} - \underbrace{T(0)}_{\text{find.}}$$

$$\int_{r=0}^{r=6400} \frac{dT}{dr}\,dr = -\frac{75}{2}\int_{r=0}^{r=6400} \frac{1}{\sqrt{6400-r}}\,dr$$

Let $u = 6400 - r$

$\Rightarrow du = -dr$.

Change Limits:

$r = 0 \Rightarrow u = 6400$

$r = 6400 \Rightarrow u = 0$.

$$-\frac{75}{2}\int_{r=0}^{r=6400} \frac{1}{\sqrt{6400-r}}\, dr = -\frac{75}{2}\int_{u=6400}^{u=0} u^{-\frac{1}{2}}\,(-du)$$

$$= -\frac{75}{2}\int_{u=0}^{u=6400} u^{-\frac{1}{2}}\, du \qquad \text{(switch limits)}$$

$$= -\frac{75}{2}\left[2u^{\frac{1}{2}}\right]_{u=0}^{u=6400}$$

$$= -75\left(\sqrt{6400} - 0\right)$$

$$= -75(80) = -6000$$

Back to original problem:

$$\int_{r=0}^{r=6400} \frac{dT}{dr}\, dr = T(6400) - T(0)$$

$$\Rightarrow \quad -6000 = 0 - T(0)$$

$$\Rightarrow \quad T(0) = 6000\,°C$$