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Lecture 12

EoL 11

Read 3.1, 3.3, 3.4
3.5.1, 3.5.2

Examples 3.9, 3.10, 3.13, 3.14
3.17, 3.18, 3.20

Exercises 3.3.1, 3.4.1, 3.5.2,
3.5.4

Today's topics:

- Limits of sequences (ct.)
- Instantaneous velocity
- Limits of functions

Example using limit rules.

Let $a_n = \dfrac{4n+5}{2n-3}$, $b_n = \dfrac{3n^2-2}{4n+9n^2}$.

Find ① $\lim_{n\to\infty} a_n b_n$, ② $\lim_{n\to\infty} \dfrac{7a_n}{b_n}$, ③ $\lim_{n\to\infty} a_n^3$.

First $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{4+\frac{5}{n}}{2-\frac{3}{n}}$ (divide by highest power)

$= \dfrac{4}{2} = 2$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{3-\frac{2}{n^2}}{\frac{4}{n}+9} = \dfrac{3}{9} = \dfrac{1}{3}$

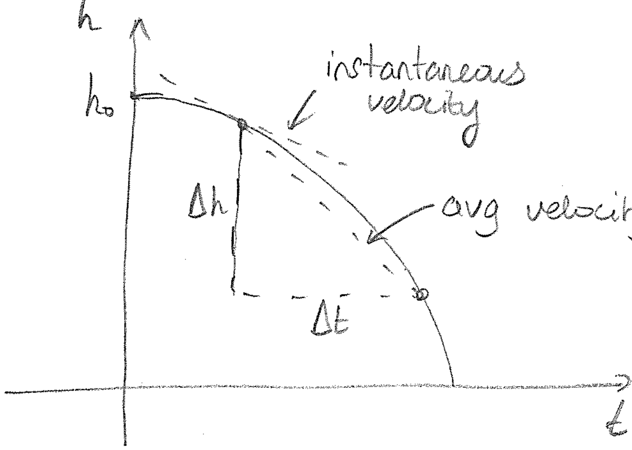
① $\lim_{n\to\infty}(a_n b_n) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$.

② $\lim_{n\to\infty}\left(\dfrac{7a_n}{b_n}\right) = 7\,\dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = 7 \cdot \dfrac{2}{1/3} = 7(6) = 42$.

③ $\lim_{n\to\infty} a_n^3 = \left(\lim_{n\to\infty} a_n\right)^3 = 2^3 = 8$. (saves time: a_n^3 messy)

Instantaneous velocity

Ball dropped from height h_0



$$h(t) = h_0 - \frac{1}{2} g t^2$$

avg velocity (slope) $= \dfrac{\Delta h}{\Delta t} = \dfrac{\text{"rise"}}{\text{run}}$

defined for $\Delta t \neq 0$

Instantaneous velocity - slope at $\Delta t = 0$.

But $\dfrac{\Delta h}{\Delta t}$ not defined at $\Delta t = 0$!

Take limits: As $\Delta t \to 0$, $v_{Avg} = \dfrac{\Delta h}{\Delta t} \to v(t)$

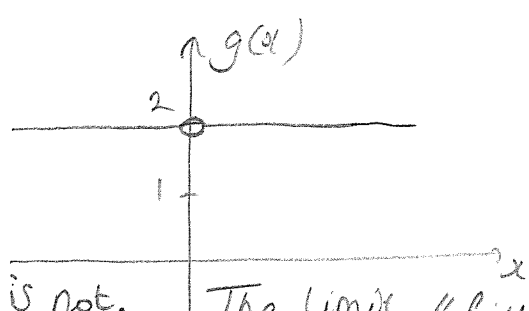
$$v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta h}{\Delta t}$$

Easier related problem.

Consider $g(x) = \dfrac{2x}{x}$. ← NOT defined at $x = 0$. $(\mathbb{D} = (-\infty, 0) \cup (0, \infty))$

For $x \neq 0$, $g(x) = 2$.

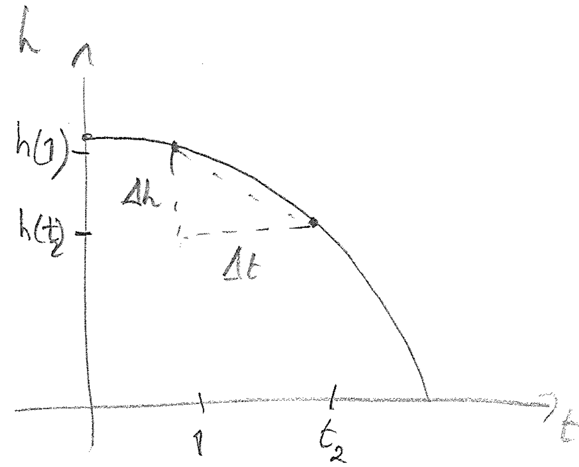
So $\lim_{x \to 0} g(x) = 2$.



The limit is defined, $g(0)$ is not. The limit "fills the gap".

Apply to velocity

$$h(t) = h_0 - \frac{1}{2} g t^2.$$



Find inst. velocity at $t=1$.

$$V_{avg} = \frac{\Delta h}{\Delta t} = \frac{h(t_2) - h(1)}{t_2 - 1}$$

$$= \frac{-\frac{1}{2} g (t_2^2 - 1)}{t_2 - 1}$$

Inst. velocity found by taking $\Delta t \to 0$ ie. $t_2 \to 1$.

$$V_{avg} = \frac{-\frac{1}{2} g (t_2 + 1)(t_2 - 1)}{t_2 - 1}$$

For $t_2 \neq 1$, $V_{avg} = -\frac{1}{2} g (t_2 + 1)$

$$\Rightarrow \lim_{t_2 \to 1} V_{avg} = -\frac{1}{2} g (1 + 1) = -g = V_{inst}.$$

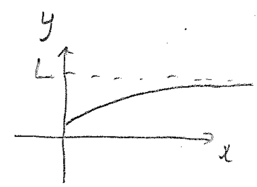
Definition: Limit of a function at a point

The limit of $f(x)$ at a point a is equal to L if as x "approaches" a, the output values, $f(x)$, approach L.

(see Ch 3.2 for more rigorous defn - beyond scope of course).

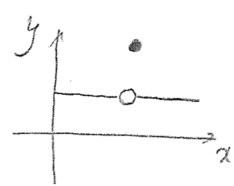
$$\lim_{x \to a} f(x) = L \quad \text{OR} \quad \text{as } x \to a,\ f(x) \to L$$

Subtle points.



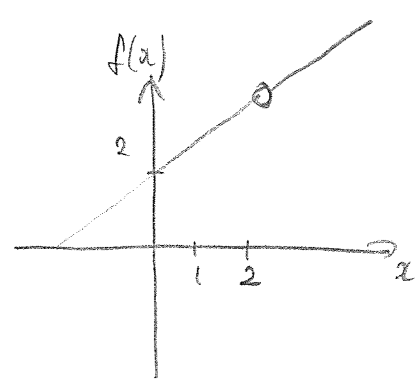
$$\lim_{x \to a} f(x) = L$$

- does not indicate that $f(x)$ ever takes the value L (though it may)



- is unaffected by f's behaviour at $x = a$.
- is shorthand for x approaching a "from both sides" (more later)

Example 1.



Let $f(x) = \dfrac{x^2 - 4}{x - 2}$.

a) Determine $\lim_{x \to 1} f(x)$.

For $x \neq 2$, $f(x) = \dfrac{(x+2)(x-2)}{x-2} = x + 2$,

$$\lim_{x \to 1} (x+2) = 3$$ "$x+2$ approaches 3 as x approaches 1"

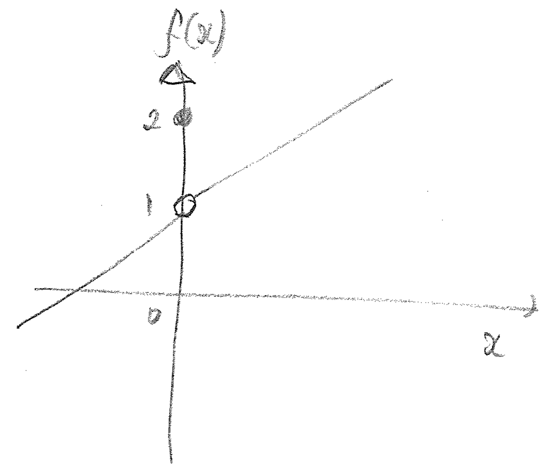
b) Determine $\lim_{x \to 2} f(x)$

x never actually reaches 2

$f(2)$ being undefined is not a problem.

$$\lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2} (x+2) = 4.$$

Example 2.



Let $f(x) = \begin{cases} x+1 & x \neq 0 \\ 2 & x = 0 \end{cases}$

What is $\lim_{x \to 0} f(x)$?

Note $f(0) = 2$. So is $\lim_{x \to 0} f(x) = 2$? No!

x gets infinitesimally close to zero, but never attains 0.

$f(x)$ looks like $x+1$ either side of limit - fill the gap.

$$\lim_{x \to 0} f(x) = 1.$$