Lecture 34.

Today's topics:

- Integration by substitution
  - Indefinite
  - Definite.

Read Ch 7.1

Ex 7.1.1 - 7.1.15

EoL 32

Project 3, EoL 29-33, Bi-weekly 6 due Monday 4.00pm.

Recall: The Chain Rule

Find the derivative of $F(x) = -\frac{1}{3}(1-x^2)^{3/2}$.

$$f(x) = F'(x) = -\underbrace{\frac{1}{3}\frac{3}{2}(1-x^2)^{\frac{1}{2}}}_{\text{outer deriv}}\ \underbrace{(-2x)}_{\text{inner deriv}}$$

evidence that chain rule has been used.

Simplify: $f(x) = x\sqrt{1-x^2}$. Now how do we find the antiderivative?

In current form, we can see that chain rule has been used. But, given

$$f(x) = x(1-x^2)^{\frac{1}{2}},$$

it is less clear how to find the antiderivative.

We use a <u>substitution</u>.

$$\int x\sqrt{1-x^2}\,dx = ?$$

Integration by substitution.

Strategy:

Define new variable (u) such that

- u is in integrand
- $\frac{du}{dx}$ is in integrand

} up to some constant multiple.

Try $u = 1-x^2$

$\frac{du}{dx} = -2x$

} both in integrand $\int x\sqrt{1-x^2}\,dx$.

Then $\int x\sqrt{1-x^2}\,dx = \int \sqrt{u}\,x\,dx$.

Next step: write entire integral in terms of u.

→ rearrange $u \Leftrightarrow x$ equations

$$\frac{du}{dx} = -2x \Rightarrow du = \underbrace{-2x\,dx}_{\text{in integral!}}$$

$$\Rightarrow x\,dx = -\frac{du}{2}$$

$$\int \sqrt{u}\,x\,dx = \int \sqrt{u}\left(-\frac{du}{2}\right) = -\frac{1}{2}\int \sqrt{u}\,du$$ (all in terms of u).

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + c$$ (now write in terms of x)

$$= -\frac{1}{3}u^{3/2} + c$$

$$= -\frac{1}{3}(1-x^2)^{3/2} + c. = F(x) + c$$

Example 2.

$$I = \int x\sqrt{x+2}\,dx.$$

$u = ?$ We don't like $\sqrt{x+2}$. $\sqrt{u}$ is easier to work with...

Try $u = x+2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$.

$$I = \int \sqrt{u}\,\underbrace{x\,dx}_{\text{write in terms of } u}$$

$x = u-2$
$dx = du$

$$\Rightarrow \quad I = \int \sqrt{u}\,(u-2)\,du$$
$$= \int (u^{3/2} - 2u^{1/2})\,du$$
$$= \frac{2}{5}u^{5/2} - 2\left(\frac{2}{3}u^{3/2}\right) + c$$
$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c.$$

Picking a substitution can be tricky....

$$I = \int \frac{x}{1+x^4}\,dx$$

Try $q = 1+x^4$

$$\frac{dq}{dx} = 4x^3 \Rightarrow dq = 4x^3\,dx \quad , \quad x\,dx = \frac{dq}{4x^2}$$

$$I = \int \frac{x\,dx}{q} = \int \frac{dq}{4x^2 q}$$

can't get rid of x

→ find another substitution.

Try $u = x^2$ (derivative is on numerator).

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \quad , \quad x\,dx = \tfrac{1}{2}du$$

$$\Rightarrow \quad I = \int \frac{x\,dx}{1+u^2} = \int \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}\int \frac{1}{1+u^2}\,du.$$

We know the antiderivative of $\frac{1}{1+u^2}$.

$$= \frac{1}{2}\arctan(u) + c$$

$$= \frac{1}{2}\arctan(x^2) + c$$

Check answer by differentiating!

$$f(x) = \tfrac{1}{2}\arctan(x^2) + c \quad , \quad f'(x) = \frac{1}{2}\,\frac{1}{1+(x^2)^2}\,(2x) = \frac{x}{1+x^4} \quad ✓$$

Substitution with Definite Integrals.

When making a substitution, we must be careful with the limits of integration.

Eg/. $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin(t)}{\cos^2(t)}\,dt$

let $u = \cos(t)$ (we see derivative also in integrand)

$\Rightarrow du = -\sin(t)\,dt$.

The limits $t=0$, $t=\frac{\pi}{6}$ are specifically for t.

Two possible approaches:

#1 Compute limits for u.

$t = 0 \Rightarrow u = \cos(0) = 1$

$t = \frac{\pi}{6} \Rightarrow u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$$\Rightarrow \int_{t=0}^{t=\frac{\pi}{6}} \frac{\sin(t)}{\cos^2(t)}\,dt \quad \int_{u=1}^{u=\frac{\sqrt{3}}{2}} \frac{-du}{u^2} = -\int_1^{\frac{\sqrt{3}}{2}} u^{-2}\,du = -\left[\frac{u^{-1}}{(-1)}\right]_1^{\frac{\sqrt{3}}{2}}$$

$$= \left[\frac{1}{u}\right]_1^{\frac{\sqrt{3}}{2}}$$

$$= \frac{2}{\sqrt{3}} - 1.$$

Note: never had to substitute t back in.

#2 Put in terms of t before using limits.

$$\int_{t=0}^{t=\frac{\pi}{6}} \frac{\sin(t)}{\cos^2(t)}\,dt = \int_{t=0}^{t=\frac{\pi}{6}} \left(-u^{-2}\right) du$$

$$= \left[u^{-1}\right]_{t=0}^{t=\frac{\pi}{6}} \qquad \text{(write in terms of } t\text{)}$$

$$= \left[\frac{1}{\cos(t)}\right]_{t=0}^{t=\frac{\pi}{6}}$$

$$= \frac{1}{\cos(\frac{\pi}{6})} - \frac{1}{\cos(0)}$$

$$= \frac{2}{\sqrt{3}} - 1$$