# Lecture 11.

Today's topics:

- Limits of sequences

EoL 10

Read 3.5

Examples 3.17, 3.18
3.20 - 3.22

Exercises 3.5.1
(a, d, e, f, g, j)

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## Limits of a sequence.

Consider sequence

$$\{a_n\}_{n=1}^{\infty} \quad \text{where} \quad a_n = 1 - \frac{1}{n}$$

$a_1 = 0$, $a_2 = 1 - \frac{1}{2} = \frac{1}{2}$, $a_3 = 1 - \frac{1}{3} = \frac{2}{3}$, $a_4 = 1 - \frac{1}{4} = \frac{3}{4}$, ....

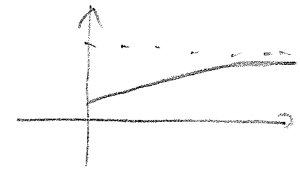
$(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots)$.

We classify sequences based on the trend in their values as $n$ increases.

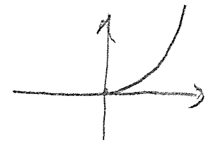
**Case 1:** **Convergent**

Sequence approaches some value $L$.

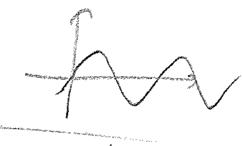
Limit of the sequence.



**Case 2:** **Divergent**



Sequence "never settles down". (no $L$ exists).



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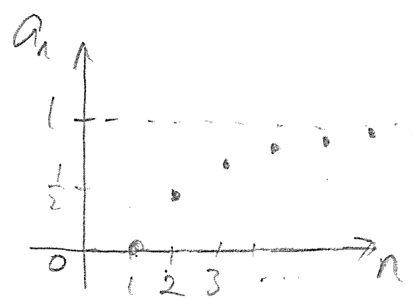
Our sequence

$$a_n = 1 - \frac{1}{n}$$




Every time $n$ increases, $\frac{1}{n}$ gets smaller.

$\Rightarrow$ $1-\frac{1}{n}$ gets closer to 1

$\Rightarrow$ $L=1$ and the sequence is convergent.

Notation: $\lim_{n\to\infty} a_n = 1$ OR $a_n \to 1$ as $n \to \infty$.

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Eg1. Find limit of sequence $a_n = \dfrac{n^2}{2n^2+1}$

Intuition: if $n$ is huge, $2n^2+1 \approx 2n^2 \Rightarrow a_n \approx \dfrac{n^2}{2n^2} = \dfrac{1}{2}$.

Formally: "divide top & bottom by highest power of $n$".

$$a_n = \frac{n^2}{2n^2+1} \times \frac{1/n^2}{1/n^2} = \frac{1}{2+\frac{1}{n^2}} \to \frac{1}{2} \text{ as } n \to \infty.$$

$$\Rightarrow \lim_{n\to\infty} a_n = \frac{1}{2}.$$

Note $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

# Divergent Sequences

No limit exists ( sequence "fails to settle down") if :

1). sequence increases without bound

Eg/ $a_n = n$. As $n \to \infty$, $a_n \to \infty$.

$\iff \lim_{n\to\infty} a_n = \infty$.

← does NOT mean limit exists - $\infty$ represents a concept.

2) decreases without bound.

Eg/. $a_n = (-1)\, 2^n$

$\lim_{n\to\infty} a_n = -\infty$ , or as $n \to \infty$, $a_n \to -\infty$.

3). sequence just never goes anywhere

Eg/. $a_n = (-1)^n$. $(-1, 1, -1, 1, \ldots)$

No single destination or growth without bound.

# Examples

Typical Qn: Determine whether the sequence converges or diverges. If it converges, determine the limit. If it diverges, describe the trend.

a) $a_n = \frac{1}{n} - \frac{1}{n+1}$

Intuition: $\frac{1}{n}, \frac{1}{n+1}$ decrease to zero - so should their difference.

$$a_n = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)} = \frac{1}{n^2+n} \to 0 \text{ as } n \to \infty.$$

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b) $a_n = -n^2 + n$

Intuition: $n^2$ grows faster than $n$ $\Rightarrow a_n \to -\infty$.

$$a_n = \underbrace{n}_{\substack{\text{big}\\+\text{ve}}}\underbrace{(1-n)}_{\substack{\text{big}\\-\text{ve}}} \Rightarrow a_n \to -\infty.$$

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c) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$

Intuition: $\sqrt{n+1} \approx \sqrt{n}$ for large $n$. dividing by large number predict $a_n \to 0$.

$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{n(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{n+1-n}{n(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \to 0 \text{ as } n \to \infty$$

$$\text{"}\frac{1}{\infty}\text{"}$$

# Limit rules (useful!)

Let $\{a_n\}$, $\{b_n\}$ be convergent sequences and $c \in \mathbb{R}$.
Then

1) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

"bring constants out of limit"

2) $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$ Eg/ $\lim_{n\to\infty} \frac{4}{n} = 4 \lim_{n\to\infty} \frac{1}{n}$

3) $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

4) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$.

## Examples

$$b_n = \frac{3n^2 - 2}{4n + 9n^2}, \quad a_n = \frac{4n+5}{2n-3}$$

a) Find $\lim_{n\to\infty} b_n$.

$$b_n = \frac{3 - \frac{2}{n^2}}{\frac{4}{n} + 9} \to \frac{3}{9} = \frac{1}{3} \quad \text{as } n\to\infty.$$

b) Find $\lim_{n\to\infty} \left(\frac{6a_n}{b_n}\right)$

$$a_n = \frac{4 + \frac{5}{n}}{2 - \frac{3}{n}} \to \frac{4}{2} = 2 \quad \text{as } n \to \infty.$$

$$\lim_{n\to\infty}\left(\frac{6a_n}{b_n}\right) = 6\,\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = 6 \cdot \frac{2}{1/3} = 36.$$