Lecture 19.

Today's topics:-
- Implicit differentiation
- Derivatives of inverse functions

Announcements:

Lab 13-16 due Friday 4pm.

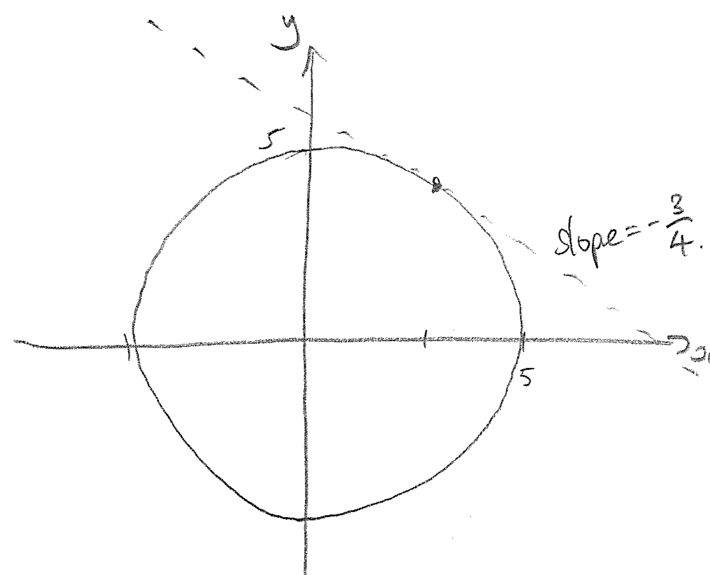
Recall:

Implicit differentiation allows us to differentiate equations that don't have the form $y = f(x)$. → Eg. $y^2 e^y = \sin(x)$.

Circle Example

$$x^2 + y^2 = 25$$

Find the slope at point $(3,4)$.



Diff. both sides with respect to x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

At $(3,4)$, $\frac{dy}{dx} = -\frac{3}{4}$

At $(\pm 5, 0)$, $\frac{dy}{dx}$ undefined

Example 2: See Lec 18.

Derivative of the natural log function

We've seen $\frac{d}{dx} \ln(x) = \frac{1}{x}$.

We can now prove this using implicit differentiation!

Set: $y = \ln(x)$. Goal: find y'.

Trick: $e^y = x$ (rearrange)

$\Rightarrow \quad e^y \frac{dy}{dx} = 1$ (differentiate implicitly)

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ (sub x back in)

And so: $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Other inverse functions

→ We found the derivative of $\ln(x)$ thanks to knowing the derivative of its inverse (e^x).

→ Can we do this for other inverse functions? (yup)

Example 1

recall $x \in [-1,1]$
$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Differentiate $f(x) = \arcsin(x)$.

hmm... we know $(\sin\theta)' = \cos\theta$.

Set $y = \arcsin(x)$

$\Rightarrow \quad \sin y = x$

$\Rightarrow \quad \cos y \, y' = 1$

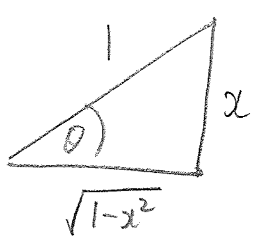
$\Rightarrow \quad y' = \dfrac{1}{\cos y}$ ← should leave answer in terms of x since given $f(\underline{\underline{x}})$.

$\Rightarrow \quad y' = \dfrac{1}{\cos(\arcsin(x))}$. ← simplify using our trig training!

$\cos(\arcsin(x))$?

and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (range of arcsin)

Set $\theta = \arcsin(x)$. $\Rightarrow x = \sin\theta$



$\Rightarrow \cos\theta = \sqrt{1-x^2}$

note could also have used $\cos^2\theta + \sin^2\theta = 1$

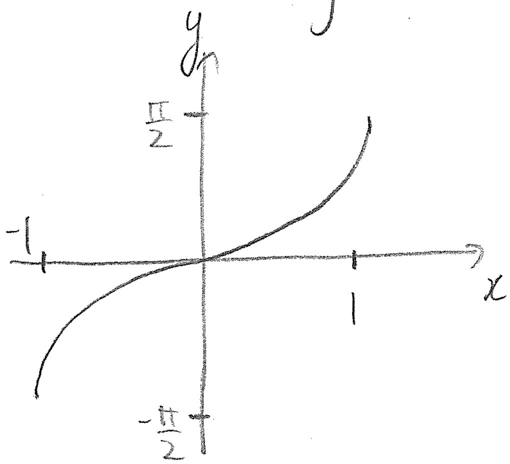
$\Rightarrow \cos(\arcsin(x)) = \sqrt{1-x^2}$

Putting it all together:

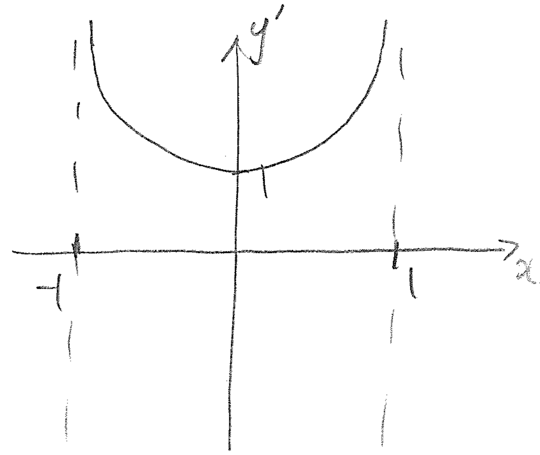
$$y' = \frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}.$$

Graphically:

$y = \arcsin(x)$



$y' = \frac{1}{\sqrt{1-x^2}}$



Example 2.

Let's do it again!

$$f(x) = \arctan(x).$$

Set $\quad y = \arctan(x)$

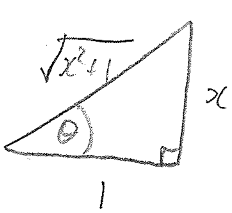
$$\Rightarrow \quad \tan y = x$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} = 1 \qquad \text{(implicit diff)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\sec^2(\arctan(x))}$$

$\sec^2(\arctan(x))$??

Let $\theta = \arctan(x)$, then $x = \tan\theta$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$$\cos\theta = \frac{1}{\sqrt{1+x^2}} \Rightarrow \sec\theta = \sqrt{1+x^2}$$

$$\Rightarrow \sec^2\theta = 1 + x^2$$

Altogether:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\arctan(x)\right) = \frac{1}{1+x^2}.$$