Lecture 30

Today's topics:

- Antiderivatives and areas
- Sigma (summation) notation

Read Ch 6.1, Examples 6.4, 6.6.

Course / instructor evaluations

http://evaluate.uwaterloo.ca.

Tutorial :- group work on optimisation & sketching

EoL 23-28 due Friday 4pm.

Antiderivatives vs. Areas

Consider a car with velocity

$$v(t) = 50 \text{ km/h}$$

and initial position $x(0) = 0$.

Recall: $x(t)$ is the antiderivative of $v(t)$, ie.

$$\frac{dx}{dt} = v$$

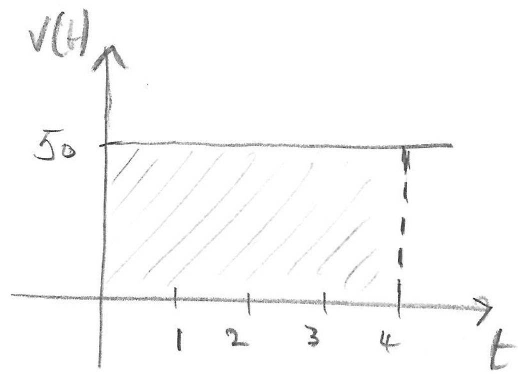
$$\Rightarrow x(t) = vt + c$$
$$= 50t$$

$(c = 0$ as $x(0) = 0)$.

How far does car travel after 4hrs?

$$x(4) = 50 \times 4 = 200 \text{ km}.$$

Graphically:



Area = base × height
$= 4 \times 50 = 200.$

Area agrees with antiderivative...

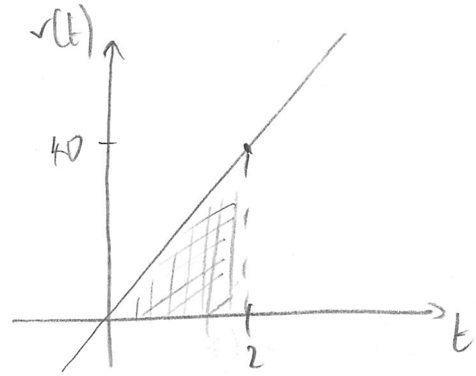
How about for

$v(t) = 20t.$

How far does car travel in 2 hours?

$x(t) = 10t^2 \quad (c=0)$

$x(2) = 10(4) = 40\,\text{km}$



$\text{Area} = \frac{1}{2}(\text{base} \times \text{height})$

$= \frac{1}{2}(2)(40) = 40.$ (agrees)

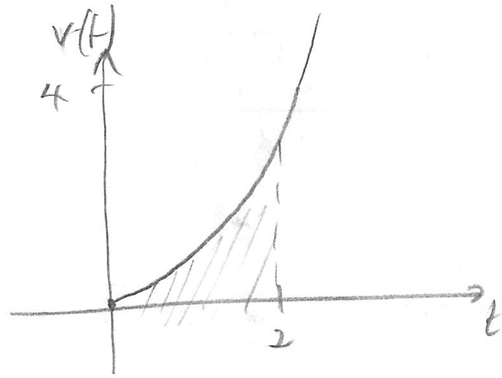
How about

$v(t) = t^2.$

How far does car travel in 2 hours?

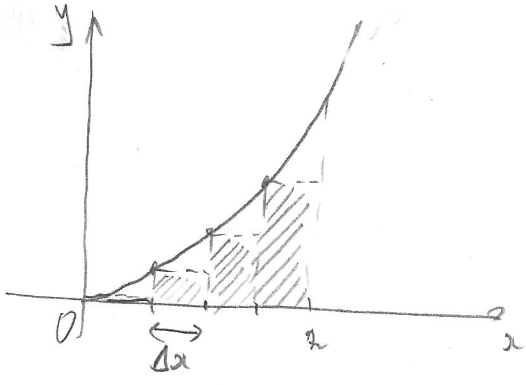
$x(t) = \frac{1}{3}t^3 \quad (c=0)$

$\Rightarrow x(2) = \frac{8}{3}\,\text{km}$



Area = ?

We like rectangles... (easy area to compute)



- The sum of rectangles is clearly an underestimate.
- But thinner rectangles get us closer to the true value.
- Mathematically

$$\lim_{\Delta x \to 0} \left(\text{area of sum of rectangles} \right) = \text{true area under curve}$$

Sigma Notation

→ We need notation for summation: (Σ upper case sigma)

upper bound (integer) → 4

$$\sum_{p=2}^{4} f(p) = f(2) + f(3) + f(4)$$

index, lower bound (integer), formula or function for obtaining series terms.

→ Put every index value (lower to upper bound, integers) into function, and sum values to give a <u>series</u>.

Examples

a) $$\sum_{i=1}^{4} \frac{i+1}{2} = \frac{1+1}{2} + \frac{2+1}{2} + \frac{3+1}{2} + \frac{4+1}{2}$$

$$= 1 + \frac{3}{2} + 2 + \frac{5}{2} = 7.$$

b) $f(x) = x^2$.

$$\sum_{i=1}^{4} f(i) = f(1) + f(2) + f(3) + f(4)$$

$$= 1^2 + 2^2 + 3^2 + 4^2$$

$$= 1 + 4 + 9 + 16$$

$$= 30$$

$\left(= \sum_{i=1}^{4} i^2 \right)$

c) $$\sum_{i=1}^{4} 10 = 10 + 10 + 10 + 10 = 40.$$

↑

$f(i) = 10$, $f(1) = f(2) = f(3) = f(4) = 10$.

Sigma notation rules.

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} (f(i)+g(i)) = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$$

$$\sum_{i=1}^{n} c \times f(i) = c \sum_{i=1}^{n} f(i) \quad , \quad c \in \mathbb{R}.$$

Eg/

$$\sum_{i=1}^{3} (6i + i^2) = \sum_{i=1}^{3} 6i + \sum_{i=1}^{3} i^2$$

$$= 6\sum_{i=1}^{3} i + \sum_{i=1}^{3} i^2$$

$$= 6(1+2+3) + (1+4+9)$$

$$= 36 + 14$$

$$= 60.$$

Two very useful sum formulae (proofs omitted)

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

"sum of first n integers"
$1+2+3+\cdots+n.$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

"sum of first n squared-integers"
$1^2+2^2+\cdots+n^2.$

Eg/

$$\sum_{i=1}^{20} 3i^2 = 3\sum_{i=1}^{20} i^2 = 3\left(\frac{20(21)(41)}{6}\right) = 10(21)(41) = 8610.$$

Caution: Formulae only work if index starts at 1.

Eg/. Compute $10+11+12+\cdots+20$. $\left(= \sum_{i=10}^{20} i \right)$.

Workaround:

$$\underbrace{\sum_{i=1}^{20} i}_{\text{formula applies}} = \underbrace{\sum_{i=1}^{9} i}_{\text{formula applies.}} + \underbrace{\sum_{i=10}^{20} i}_{\text{what we want to find.}}$$

$$\Rightarrow \sum_{i=10}^{20} i = \sum_{i=1}^{20} i - \sum_{i=1}^{9} i$$

$$= \tfrac{1}{2}20(21) - \tfrac{1}{2}9(10)$$

$$= 210 - 45$$

$$= 165$$

Next time: Apply this summation theory to the summation of many rectangles under a curve.