Lecture 7

Today's topics:

Logarithmic functions.

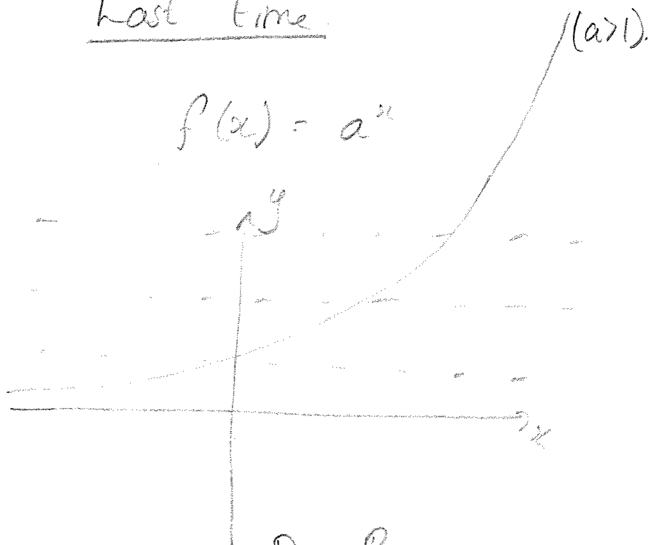
- laws, graphing, solving eqns, models.

Read Ch 2.5
Examples 2.20 - 2.21
Exercise 2.5.4 - 2.5.8
2.8.8 - 2.8.9.
EoL 7
Bi-weekly A1
→ due 28 Sept

Last time

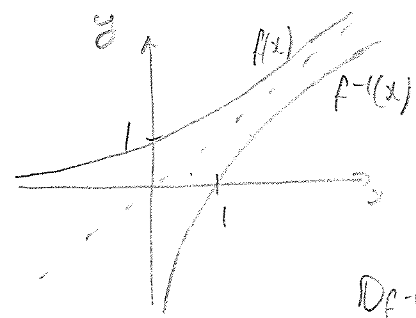
$f(x) = a^x$

HLT.



$\mathbb{D}_f = \mathbb{R}$
$\mathbb{E}_f = \{x \in \mathbb{R} : x > 0\}$

One-to-one? Yes.
$\Rightarrow f^{-1}(x)$ exist.



$\mathbb{D}_{f^{-1}} = \{x \in \mathbb{R} : x > 0\}$
$\mathbb{E}_{f^{-1}} = \mathbb{R}$

Inverse of an exponential function

... is a logarithmic function.

$$\underset{\text{base}}{y = a^{x}} \Longleftrightarrow x = \log_a y \qquad (a > 0,\ a \neq 1)$$

exponent; logarithm base a.

"x is the exponent to which a must be raised to obtain y"

Examples.

a) $\log_2(32)$ - what power of 2 gives 32 ?

Let $y = \log_2(32)$. Then $2^y = 32 \Rightarrow y = 5$.

So $\log_2(32) = 5$.

b) $\log_{10}(1000)$

Let $y = \log_{10}(1000)$. Then $10^y = 1000 \Rightarrow y = 3$.

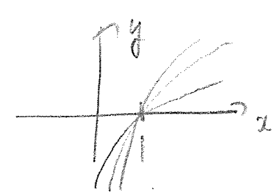
So $\log_{10}(1000) = 3$.

Useful identities

$\log_b(b^x) = x$ (for all $x \in \mathbb{R}$)

$b^{\log_b(x)} = x$ (for all $x > 0$)

$\log_b(1) = 0$ all log functions go through $(1,0)$



Logarithm Laws (valid for $x, y > 0$)

1. $\log_b(xy) = \log_b(x) + \log_b(y)$

2. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

3. $\log_b(x^n) = n\log_b(x)$

$$\log_b(x^3) = \log_b(xxx)$$
$$= \log_b x + \log_b x + \log_b x$$
$$= 3\log_b x.$$

Beware:

$$\log_b(x+y) \neq \log_b x + \log_b y$$

Eg/ $\log_b(2) = \log_b(1+1) \neq 2\log_b(1) = 0$.

Use powers of 2 to check laws (help remember)

Eg/ $\log_2(8) = 3$.

1. $\log_2(4\times2) = \log_2 4 + \log_2 2 = 2 + 1 = 3$ ✓
2. $\log_2\left(\frac{16}{2}\right) = \log_2 16 - \log_2 2 = 4 - 1 = 3$ ✓
3. $\log_2(2^3) = 3\log_2(2) = 3\times1 = 3$ ✓

The natural logarithm

Logarithm base e is the inverse of e^x

$$y = \log_e(x) \iff x = e^y.$$

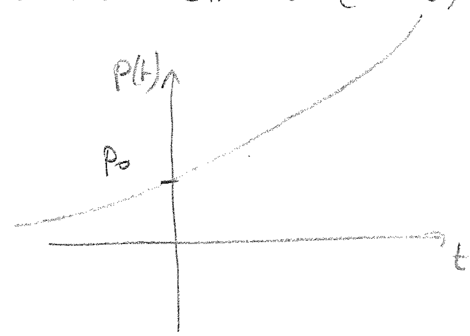
Notation: $\log_e(x) = \ln(x)$

Modelling Example.

$P(t)$ - size of bacteria population at time t. (hours)

$P(t) = P_0 \, 2^{t/3}$

Initial pop. size.



How many cells after 9 hrs?

$P(9) = P_0 \, 2^{9/3} = 8P_0 \rightarrow$ 8 times what we started with.

Find t given P? Requires logs!

Given population of 2000, how long before it reaches 10 000?

Set $P_0 = 2000$, $P = 10\,000$.

$$10\,000 = 2000 \times 2^{t/3}$$

$$\Rightarrow 2^{t/3} = 5$$

$$\Rightarrow \log_2\left(2^{t/3}\right) = \log_2 5$$

$$\Rightarrow t/3 = \log_2 5$$

$$\Rightarrow t = 3\log_2 5 \simeq 6.97 \text{ hours}$$

Estimate without calculator?

$$\log_2 4 < \log_2 5 < \log_2 8$$

$$2 < \log_2 5 < 3 \quad \Rightarrow$$

$$6\text{hrs} < t < 9\text{hrs}$$

Changing bases - exponential function

Convenient to use base e. (differentiation purposes)

$$y = b^x$$

"ln both sides" $\quad \ln y = \ln(b^x) = x\ln(b) \quad$ (law 3)

"$e^{(\cdot)}$ both sides" $\quad e^{\ln y} = e^{x\ln(b)}$

$$\Rightarrow \quad y = e^{x\ln(b)} \qquad \therefore \quad \boxed{\underbrace{b^x}_{\text{base } b} = \underbrace{e^{x\ln(b)}}_{\text{base } e}}$$

Changing bases - logarithmic functions.

Similar argument: $y = \log_b(x)$

$$\Rightarrow \quad b^y = x$$

$$\Rightarrow \quad \ln(b^y) = \ln(x)$$

$$\Rightarrow \quad y\ln(b) = \ln(x)$$

$$\Rightarrow \quad y = \frac{\ln(x)}{\ln(b)} \qquad \therefore \quad \boxed{\overbrace{\log_b(x)}^{\text{base } b} = \overbrace{\frac{\ln(x)}{\ln(b)}}^{\text{base } e}}$$

Eg/ $\quad \log_7(20) = \dfrac{\ln(20)}{\ln(7)}$

$$3^{\sin(x)} = e^{\sin(x)\ln(3)}$$