Lecture 26

Today's topics:

- Extreme values
  - technical definitions
  - extreme value theorem
  - finding global extrema with the 'closed interval method'

Read Ch 5.2

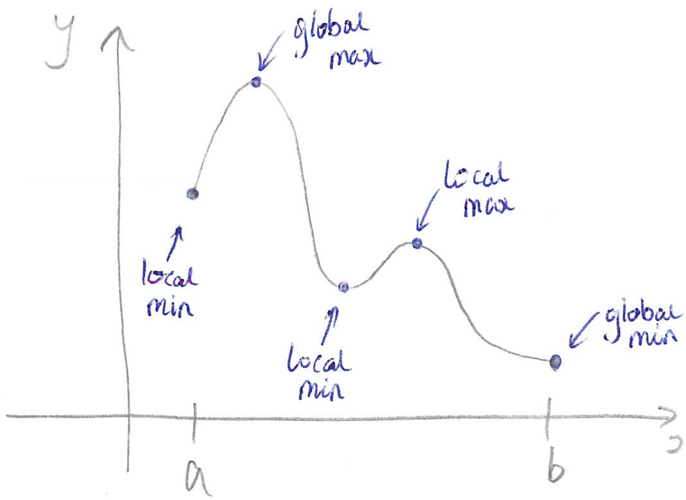
Ex 5.2.1 - 5.2.9
5.2.16 - 5.2.22

EoL 23

EoL 17-22 due today 4pm.

Monday's tutorial - Lesson on Shapes of graphs.

Maximum and minimum values : two flavours.



- $f(x)$ on the interval $[a,b]$.
- maxima & minima are collectively referred to as extrema.

Technical Definitions.

Let $c \in \mathbb{D}_f$ (a number from domain of f).
Then, $f(c)$ is a

- global / absolute maximum if $f(x) \le f(c)$ for all $x \in \mathbb{D}_f$
- local maximum if $f(x) \le f(c)$ for x 'near' (local) to c.

Swap $(\le)$ for $(\geq)$ to obtain minima definitions.

Subtle points

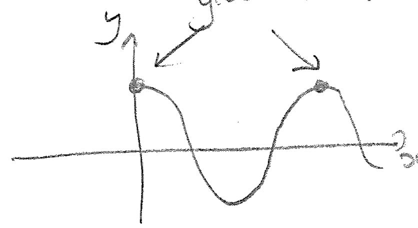
(but always call it global).

- a global max is also a local max.
- global max does not have to be unique

Eg $y = \cos(x)$

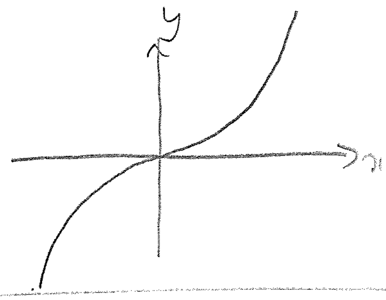
(and local)

global maxima



- global max need not exist

Eg $y = x^3$



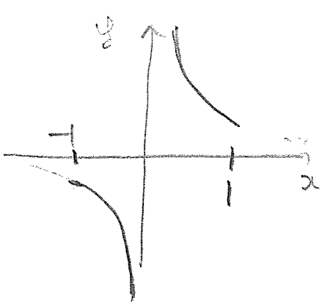
(∞ is not a number).

When do global max / min exist?

Extreme Value Theorem.

A continuous function defined on a bounded, closed interval always achieves both a global max and a global min.

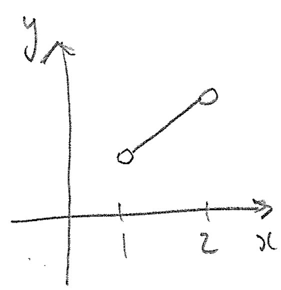
Why continuous?



$y = \frac{1}{x}$ on $[-1, 1]$.

no global min/max.

Why closed?

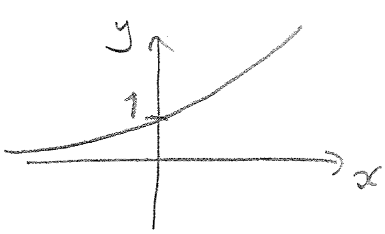


$y = x$ on $(1, 2)$.

no global min / max.

What is the largest number less than 1?? DNE.

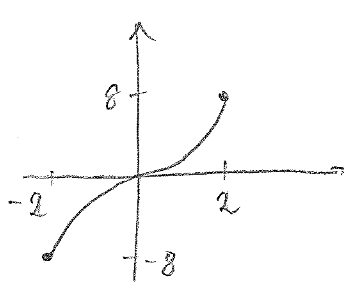
Why bounded?



$y = e^x$ on $\mathbb{R}$.

no global min / max.

Satisfies all 3



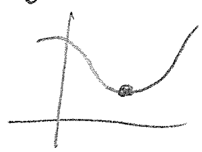
$y = x^3$ on $[-2, 2]$

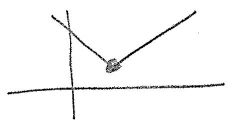
glob max $= 8$

glob min $= -8$.

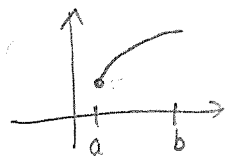
Locating Extreme Values

An extreme value c must satisfy one of the following:

1. $f'(c)=0$ 

2. $f'(c)$ DNE 

In these cases, c is called a critical number

3. c is an endpoint of interval 

The "Closed Interval Method" (CIM)

→ algorithm for finding global extrema

Let f be continuous over a closed, bounded interval $I=[a,b]$. Then:

1. Find f'
2. Determine all critical points in I
3. Evaluate f at all critical pts AND end pts a, b.
4. Compare. Largest value - global max.
   Smallest value - global min.

Example:

$$f(x) = -x^2(x-4)(x+4) \quad \text{on interval } I = [-2, 4].$$

Can we apply CIM?

continuous? ✓: $f(x)$ is a polynomial

interval bounded and closed? ✓ $I = [-2,4]$.

1. $$f(x) = -x^2(x^2-16)$$
$$= -x^4 + 16x^2$$
$$\Rightarrow f'(x) = -4x^3 + 32x$$
$$= 4x(-x^2+8)$$

2. Critical points:

$f'(x) = 0 \Rightarrow 4x(-x^2+8) = 0$

$\Rightarrow x = 0, \pm 2\sqrt{2}$

Note: $-2\sqrt{2}$ out of interval: ignore!

$f'(x)$ undefined anywhere? nope.

3. Evaluate f at:

end points

$f(-2) = -4(-6)(2) = 48$

$f(4) = 0$

critical points

$f(0) = 0$

$$f(2\sqrt{2}) = -8(2\sqrt{2}-4)(2\sqrt{2}+4)$$
$$= -8(8-16)$$
$$= 64.$$

4. Compare:

$f(2\sqrt{2}) = 64$ - global max

$f(0) = f(4) = 0$ - global min.

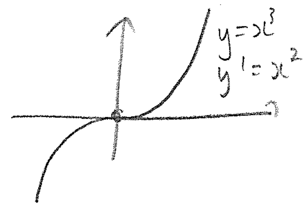
Elaborating on local extrema

→ Information from f' can tell us whether critical points are associated with local extrema.

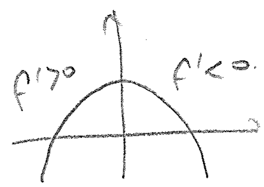
First derivative test

Let c be a critical number ($f'(c)=0$ or $f'(c)$ DNE)

a) f' has same sign either side of c
→ neither max nor min.



b) f' +ve on left
-ve on right
→ local max at c.



c) f' -ve on left
f' +ve on right
→ local min at c