Lecture 14.

Today's topics:

- continuity of functions.

Read Ch 3.7

Ex. 3.7.2, 3.7.6.

Announcements.

→ Midterm review Sat 4-6pm, MC 1056.

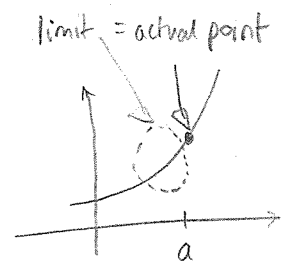
→ EoL 8-11 due today (12pm)

→ Prepare cheat sheet.

Learning Objectives

- State defn of continuity, explain concept intuitively.
- Identify continuous / discontinuous functions.
- Compute parameters that make a function continuous.
- State the Intermediate Value Theorem (IVT)
- Use IVT to determine existence of solutions.

Definition: continuity at a point.



A function f is continuous at the point a if

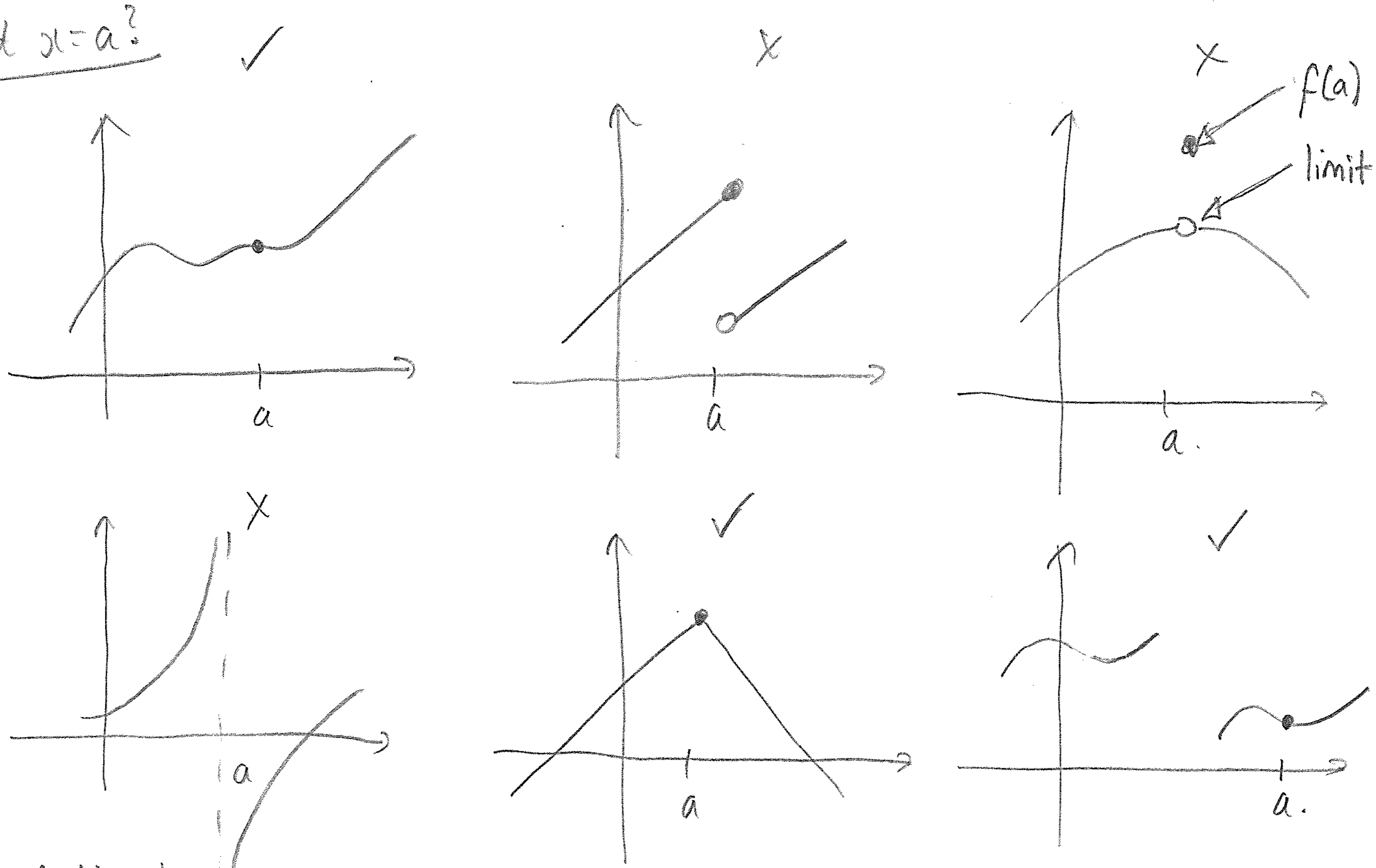
$$\lim_{x \to a} f(x) = f(a).$$

"f takes value we expect at a"

Means 3 things:

1) $\lim_{x \to a} f(x) = L$ exists
2) $f(a)$ is defined
3) $L = f(a)$

Cont. at $x=a$?

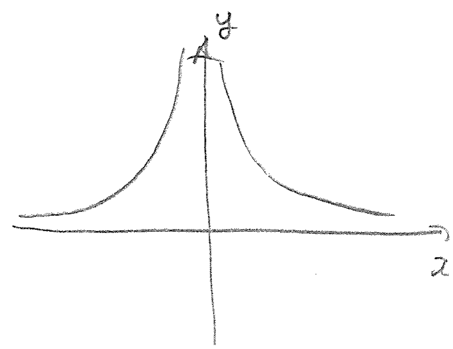


Rule of thumb: can we draw $f(x)$ through $x=a$ without lifting pen? If yes, $f(x)$ is continuous.

Definition: Continuous function

A function f is called continuous if it is continuous on all points in its domain.

Eg. $f(x) = \frac{1}{x^2}$



$\mathbb{D} = (-\infty, 0) \cup (0, \infty)$

There is an infinite discontinuity at $x=0$.
But $0 \notin \mathbb{D}$.
f is continuous on $(-\infty, 0)$ and $(0, \infty)$
$\Rightarrow$ f is a continuous function.

Examples of continuous functions.

- polynomials
- trig functions
- rational functions
- log functions
- root functions
- exponential functions

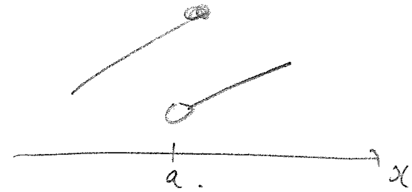
Combinations of functions.

- If f, g continuous on $[a, b]$ then $f+g$, fg, $\frac{f}{g}$ $(g \neq 0)$ are continuous. Eg/. $\frac{e^x \sin x}{x^2}$ continuous.
- If g is cont. at a, and f is cont. at $g(a)$, then $f(g(a))$ is cont. at a.

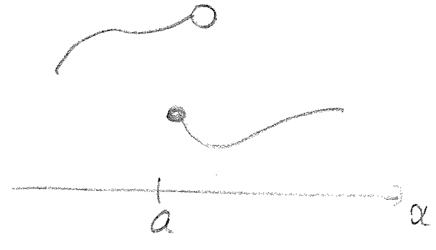
Eg/. e^{e^x} continuous.
$\ulcorner f(x) = e^x,\ g(x) = e^x \lrcorner$

Left and Right Continuity

$f(x)$ is cont. from the left if
$$\lim_{x \to a^-} f(x) = f(a).$$

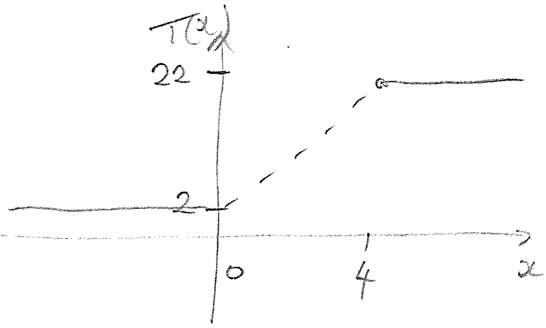


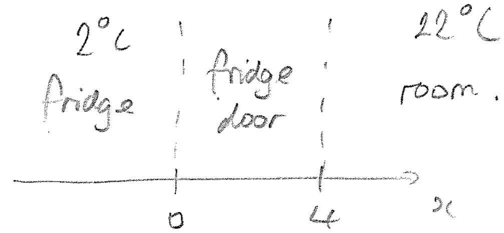
and cont. from right if
$$\lim_{x \to a^+} f(x) = f(a)$$



Exercise: Computing parameters to make function continuous.

$T(x)$ temperature at position x





Assume linear change of $T(x)$ through door.

$$T(x) = \begin{cases} 2 & x < 0 \\ ax+b & 0 \leq x < 4 \\ 22 & x \geq 4. \end{cases}$$

Goal: Choose a, b to make $T(x)$ continuous.

By construction, $T(x)$ cont. on $(-\infty, 0)$, $(0, 4)$, $(4, \infty)$.
Focus on $x = 0$, $x = 4$.

$x = 0$.

$$\left.\begin{aligned} \lim_{x \to 0^-} T(x) &= 2 \\ \lim_{x \to 0^+} T(x) &= b \end{aligned}\right\} \quad b = 2.$$

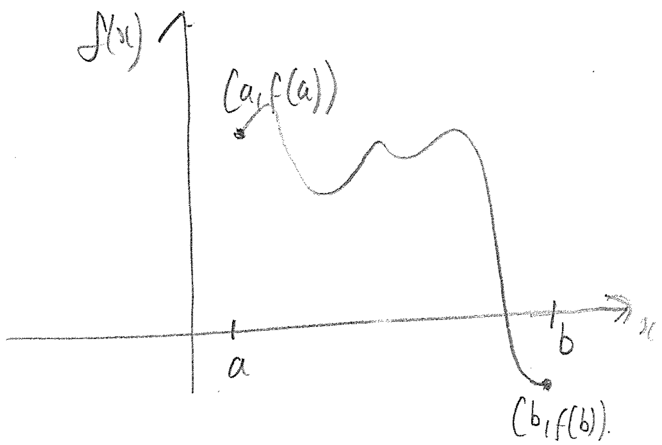
(left and right limits must be equal for continuity.)

$x = 4$

$$\left.\begin{aligned} \lim_{x \to 4^-} T(x) &= 4a + b \\ \lim_{x \to 4^+} T(x) &= 22 \end{aligned}\right\} \quad a = 5$$

$\therefore$ $a = 5$, $b = 2$ makes $T(x)$ continuous.

A thought experiment



Suppose f is cont. on $[a,b]$,

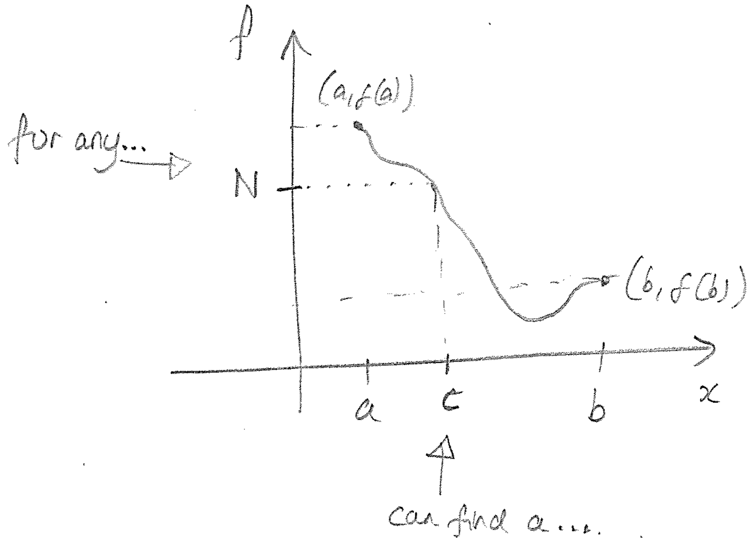
and $f(a) > 0$, $f(b) < 0$.

Must f have a root in $[a,b]$?

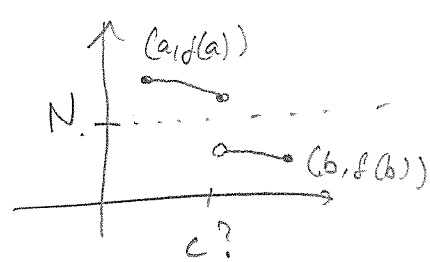
YES! (cannot lift pen).
(very useful).

More formally: The Intermediate Value Theorem (IVT)

Suppose f is cont. on $[a,b]$. Let N be any number between $f(a)$ and $f(b)$. Then there exists at least one value $c \in (a,b)$ such that $f(c) = N$.



Note: f must be continuous!



How can IVT help us?

Solve $e^x - x^2 = 0$. $x = ?$

Cannot do algebraically!

Does a solution exist? Use IVT.

Let $f(x) = e^x - x^2$.

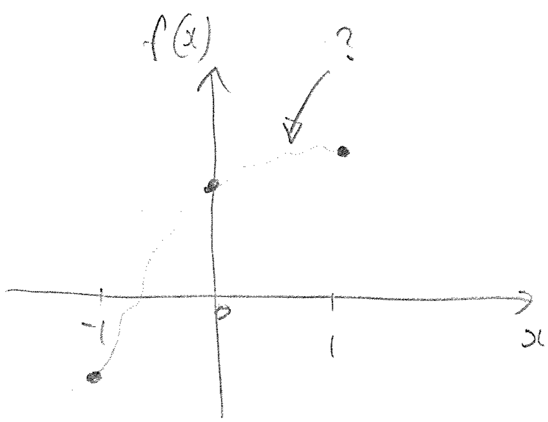
Q/ Does there exist a c such that $f(c) = 0$?

Try values:

$f(1) = e - 1 > 0$

$f(0) = 1 - 0 > 0$.

$f(-1) = \frac{1}{e} - 1 < 0$ (!)



$f(x)$ continuous.

$f(1) > 0$, $f(-1) < 0$

By IVT there is a $c \in (-1, 1)$ such that $f(c) = 0$.

ie. a solution exists!