Lecture 3

Today's topics

- Properties of functions
  - even, odd, monotonic, periodic
- Sequences - notation
- Combinations of functions

Is a function even, odd, or neither?

$$f(x) = x^4 - x^2 + 1$$

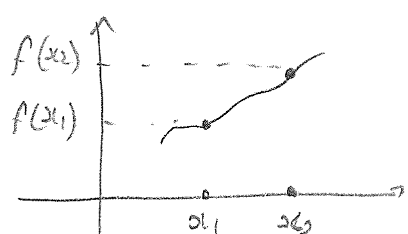
Even: $f(-x) = f(x)$
Odd: $f(-x) = -f(x)$

Plan: sub in "$-x$" to argument of f.

$(ab)^n = a^n b^n$

$$f(-x) = (-x)^4 - (-x)^2 + 1$$
$$= (-1)^4(x)^4 - (-1)^2(x)^2 + 1$$
$$= x^4 - x^2 + 1$$
$$= f(x) \qquad \Rightarrow \quad f \text{ is even}$$

Monotonic functions



- $f(x)$ is increasing on interval if $f(x_2) > f(x_1)$ for all pairs $x_2 > x_1$
- similar for decreasing
- if $f(x)$ is increasing / decreasing on $\mathbb{D}$, $f(x)$ is monotonic.

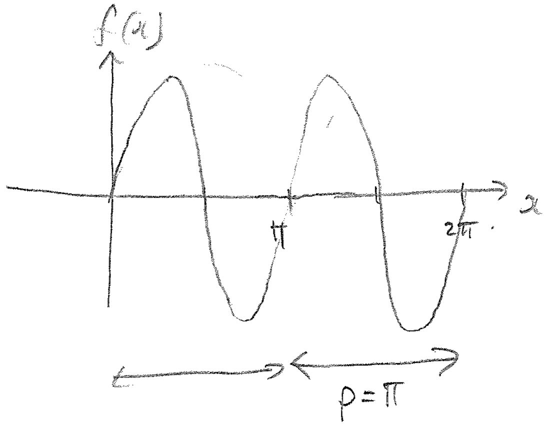
Eg/ $f(x) = x$ ✓
$f(x) = x^2$ ✗

Periodic functions (with period p)

$f(x+p) = f(x)$ for all $x \in \mathbb{R}$

p: smallest +ve value such that this holds.

Eg/ $f(x) = \sin(2x)$



$$\begin{aligned} f(x+\pi) &= \sin(2(x+\pi)) \\ &= \sin(2x+2\pi) \\ &= \sin(2x) \\ &= f(x) \end{aligned}$$

Sequences

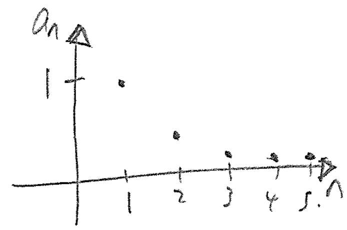
A sequence is a list of numbers with <u>definite order</u> (unlike sets).

Notation $\{a_n\}_{n=1}^{n=M} = (a_1, a_2, \ldots, a_M)$

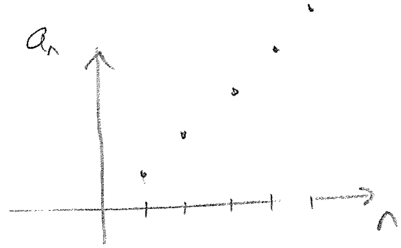
Eg/ $\left\{\frac{1}{n^2}\right\}_{n=1}^{n=5} = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}\right)$

$a_n = \frac{1}{n^2}$, $a_3 = \frac{1}{9}$

Graph:



Eg. $(2, 4, 6, 8, 10, \ldots\ldots) = \{2n\}_{n=1}^{\infty}$



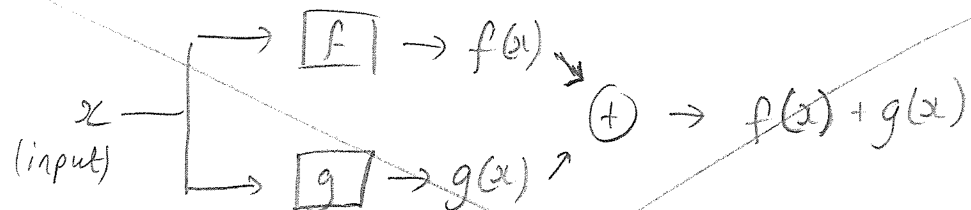
$a_n = 2n.$

Combining functions

Notation:

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(fg)(x) = f(x)g(x)$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0.$

Computing output:



Input is fed into each function independently, then function outputs are combined.

Finding the domain:

Both f and g must be defined at x.

Eg/ $f(x) = \sqrt{2-x}$, $g(x) = \sqrt{x-1}$

a) Domain of f?

$2-x \geq 0 \Rightarrow x \leq 2.$ $\mathbb{D}_f = (-\infty, 2]$

b) Domain of g?

$x-1 \geq 0 \Rightarrow x \geq 1.$ $\mathbb{D}_g = [1, \infty)$

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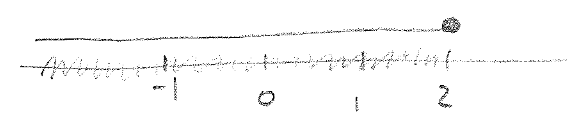
c) Domain of $f+g$?



$\mathbb{D}_f$

$\mathbb{D}_g$

Overlap $[1,2]$. $\mathbb{D}_{f+g} = \mathbb{D}_f \cap \mathbb{D}_g = [1,2].$

~~Read Ch 2.2.2.~~
Examples 2.8, 2.9
Exercises 2.8.4, 2.8.5.

d) Domain of $\dfrac{f}{g}$?

f, g both defined on $[1,2]$

Now need $g(x)$ non-zero.

$g(x) = 0 \Rightarrow \sqrt{x-1} = 0 \Rightarrow x-1 = 0 \Rightarrow x = 1.$

Exclude $x = 1$

$$\mathbb{D}_{\frac{f}{g}} = (1, 2].$$

Composition of Functions

$$\underset{\text{(input)}}{x} \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x))$$

$$f \circ g(x) = f(g(x)).$$

Eg/. $f(x) = x^2 + 1, \quad g(x) = \frac{1}{x}.$

$$f \circ g(x) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 + 1 = \frac{1}{x^2} + 1$$

$$g \circ f(x) = g(x^2 + 1) = \frac{1}{x^2 + 1}$$

Domain of composite functions

$$f(\underbrace{g(x)})$$

- g must be defined at x
- f must be defined at $g(x)$.

Eg/. $g(x) = \sqrt{2x+1}, \quad f(x) = \frac{1}{x-2}$

Domain of $f \circ g$?

$\mathbb{D}_g = [-\frac{1}{2}, \infty)$. $\quad \mathbb{D}_f = \{x \in \mathbb{R} : x \neq 2\}$

$f \circ g = \frac{1}{\sqrt{2x+1} - 2}$ - not defined when

$\sqrt{2x+1} = 2 \Rightarrow 2x + 1 = 4 \Rightarrow x = \frac{3}{2}$

$$\mathbb{D}_{f \circ g} = \left\{x \in \mathbb{R} : x \geq -\tfrac{1}{2}, x \neq \tfrac{3}{2}\right\}$$
$$= [-\tfrac{1}{2}, \tfrac{3}{2}) \cup (\tfrac{3}{2}, \infty)$$

Read Ch 2.2.2
Examples 2.8, 2.9
Exercises 2.2.3
2.8.4 - 2.8.6