Lecture 25

Today's topics:

- Indeterminate forms (2)
- Extreme values

Read Ch 5.2.1

Ex 5.2.1 – 5.2.9

(computing extreme values)

EoL 22 (indet forms)

Project 2 – grade rubric.
- deadline Sunday midnight
- piazza

Recall: L'Hopital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} \overset{(H)}{=} \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$\lim \frac{f}{g}$ has form "$\frac{0}{0}$" OR "$\frac{\infty}{\infty}$"

indeterminate forms.

Eg/ $\lim_{x \to 0} \frac{\tan(3x)}{\sin(2x)}$ $\left(\frac{0}{0}\right)$

$$\overset{(H)}{=} \lim_{x \to 0} \frac{3\sec^2(3x)}{2\cos(2x)}$$

$$= \frac{3\sec^2(0)}{2\cos(0)}$$

$$= \frac{3}{2}.$$

(note: drop limit sign when you choose to process the limit).

Exponentials vs. Polynomials

Compute $\lim_{x\to\infty} \frac{x^2}{e^x}$ $\left(\frac{\infty}{\infty}\right)$

$\overset{(H)}{=} \lim_{x\to\infty} \frac{2x}{e^x}$ $\left(\frac{\infty}{\infty}\right)$ – use l'Hopital again!

$\overset{(H)}{=} \lim_{x\to\infty} \frac{2}{e^x}$

$= 0$ (exponential wins).

How about

$$\lim_{x\to\infty} \frac{1000x^{1000}}{e^x} \quad ?$$

$= 0$ by repeated l'Hopital

(exponentials always outpace polynomials as $x\to\infty$)

Other indeterminate forms?

Note: bad things happen when we treat ∞ as a number.

Eg. $2\infty = \infty$

$\infty + \infty = \infty$

$\Rightarrow \infty = 0$??

Combinations of 0 and ∞:

	$\frac{0}{0}$	$\frac{\infty}{\infty}$	$\frac{0}{\infty}$	$\frac{\infty}{0}$	$\infty \cdot 0$	$\infty \cdot \infty$	$0 \cdot 0$	$\infty - \infty$	$\infty + \infty$
indeterminate?	✓	✓	× $(=0)$	× $(=\infty)$	✓	× $(=\infty)$	× $(=0)$	✓	× $(=\infty)$

Eg $\lim_{x\to\infty} \frac{e^{-x}}{x}$ Eg $\lim_{x\to 0} \frac{1/x}{x^2}$ Eg $\lim_{x\to\infty} xe^{x}$.

There are more when we consider exponents... Eg. 0^0, 1^∞, ∞^0 ← (beyond M127)

Indeterminate Products. $(0.\infty)$

Eg $\lim_{t\to\infty} t\,e^{-t}$ (form $0.\infty$)

→ Trick: write as a fraction by using the reciprocal of one of the factors.

$$\lim_{t\to\infty} \frac{t}{(1/e^{-t})} = \lim_{t\to\infty} \frac{t}{e^t}$$ ← now L'Hopital applies! $\left(\frac{\infty}{\infty}\right)$

$$\overset{(H)}{=} \lim_{t\to\infty} \frac{1}{e^t} = 0$$

Note: we could have chosen to use reciprocal of t:

$$\lim_{t\to\infty} t e^{-t} = \lim_{t\to\infty} \frac{e^{-t}}{(1/t)} \qquad (\text{form } \tfrac{0}{0})$$

$$\overset{(H)}{=} \lim_{t\to\infty} \frac{-e^{-t}}{-(1/t^2)} \qquad \ldots \text{ getting harder to evaluate}$$

Rule of thumb: choose form with derivatives that simplify.

Eg $(t)' = 1$ (yay)

$\left(\frac{1}{t}\right)' = -\frac{1}{t^2}$ (boo.)

Indeterminate Differences ($\infty - \infty$)

Eg/ $$\lim_{x\to\infty} (2x - x) \qquad (\text{form } \infty - \infty)$$

$$= \lim_{x\to\infty} x = \infty$$

Eg/ $$\lim_{x\to 0^+} \left[\frac{1}{\sin x} - \frac{1}{x^2}\right]$$

Separately: $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$, $\lim_{x\to 0^+} \frac{1}{\sin(x)} = \infty$

Trick: bring terms together as a fraction (L'Hopi??)

$$\lim_{x\to 0^+} \left(\frac{-\sin(x) + x^2}{x^2 \sin(x)}\right) \qquad (\text{form } \tfrac{0}{0}).$$

common denominator.

$$\overset{(H)}{=} \lim_{x\to 0^+} \frac{-\cos(x) + 2x}{2x\sin(x) + x^2\cos(x)} \qquad \text{form } \left(\frac{-1}{0}\right)$$

$$= -\infty$$

In summary:

- forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, use L'Hopital
- forms $0.\infty$, $\infty - \infty$, write as a fraction & use L'Hopital.

Extra example:

a) $\lim_{x\to 0^+} x\ln(x)$ (form $0.(-\infty)$)

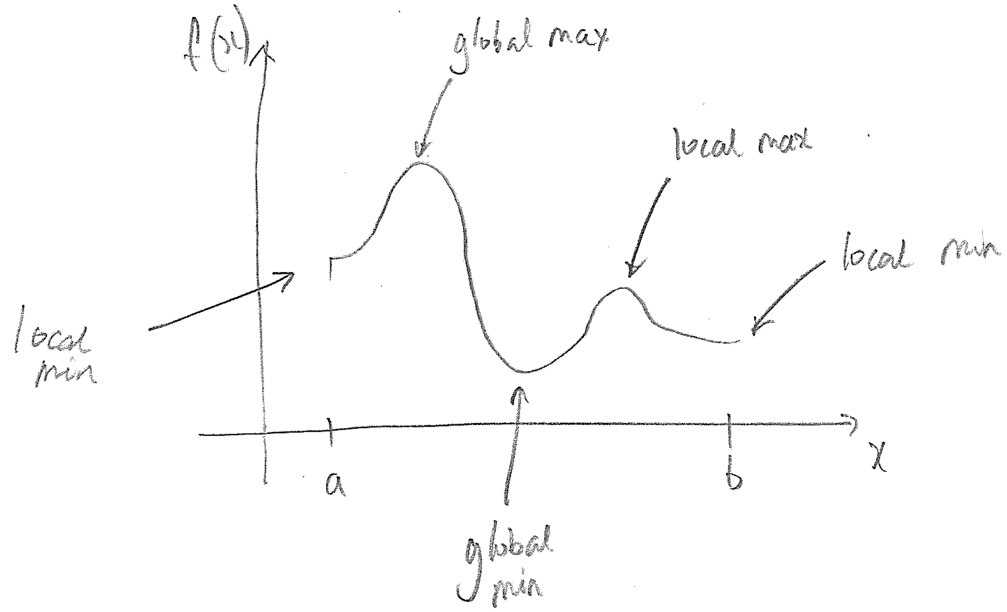
$$= \lim_{x\to 0^+} \frac{\ln(x)}{\left(\frac{1}{x}\right)}$$

$$\overset{(H)}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x\to 0^+} \frac{x}{(-1)}$$ (multiply top & bottom by x^2)

$$= 0$$

Next time: Extreme values.



How do we find these max/min values?

differentiation