Lecture 20

Today's topics:-

- Related rates
- Modelling using derivatives

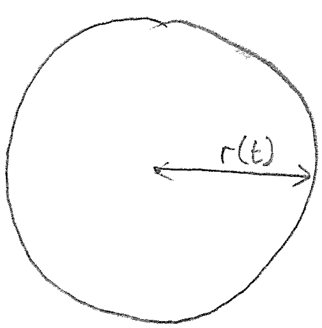
- EoL 18
- Watch Mobius lec 18
- Project 2
  - due Fri 2 Nov.
- Ex 5.1.2 (rel. rates q).

Related Rates.

→ Often, multiple related quantities change as a function of the same variable.

→ Use implicit differentiation to find related rates of change

Eg/



Radius of sphere changes at rate 2 m/s.

How quickly is volume changing at $r = 1\text{m}$.

We know $\dfrac{dr}{dt} = 2$, $V(t) = \dfrac{4}{3}\pi\left[r(t)\right]^3$

V & r are related quantities that both change in time, t.

Find $\dfrac{dV}{dt}$.

Implicit diff: $\dfrac{dV}{dt} = \dfrac{4}{3}\pi\left(3r^2\dfrac{dr}{dt}\right)$

$= 4\pi r^2 (2) = 8\pi r^2$

At $r=1$, $\frac{dV}{dt} = 8\pi \left(m^3/s\right)$ ← include units if asked for them.

Eg/. The ideal gas law is

$$PV = nRT$$

P - pressure
V - volume
n - # moles of gas
R - constant
T - temperature..

Assume fixed volume: $V = 10L$.
n, R constant.

Gas heated at rate $5K/h$ $\left(\frac{dT}{dt} = 5K/h\right)$

Find rate of pressure increase.

$$\frac{d}{dt}(PV) = \frac{d}{dt}(nRT)$$

$$\Rightarrow P'V + PV' = nRT'$$

$V' = 0$ (V const.)
$T' = 5K/h$

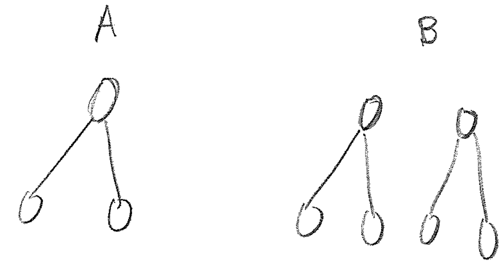
$$\Rightarrow P'(10L) = nR(5K/h)$$

$$\Rightarrow P' = \frac{nR}{2} K/hL$$

Modelling using derivatives

→ Often more intuitive to set up model using derivatives.

Eg1. Simple population growth:
- assume the growth rate is proportional to the population size.



"twice the pop. size, twice the # offspring".

Then $\dfrac{dP}{dt} \propto P$

ie. $\dfrac{dP}{dt} = kP$, k constant

$P(t)$ - population size at time t.

Definition: Differential equation (DE)

An equation involving an (unknown) function and one or more of its derivatives.

$\dfrac{dP}{dt} = kP$: differential equation involving unknown function $P(t)$ and its derivative.

→ A solution to the DE is a function $P(t)$ that satisfies the given equation.

→ Finding a solution can be tricky, but checking a proposed soln is straightforward.

Ex/ Show that $y = e^x$ is a soln to the DE

$$y' = y.$$

Let $y = e^x$.

Then $y' = e^x \Rightarrow y' = y$ and so $y = e^x$ is a soln

Ex/ Show that $P(t) = P_0 e^{kt}$ is a soln to

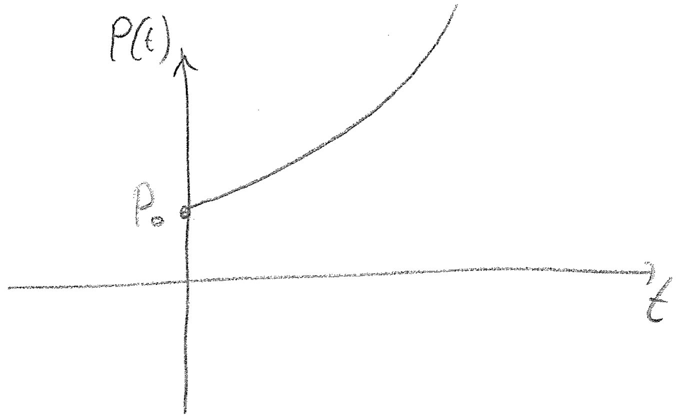
$$\frac{dP}{dt} = kP.$$

$$\frac{d}{dt} P(t) = \frac{d}{dt}\left[P_0 e^{kt} \right] = P_0 \frac{d}{dt}\left(e^{kt}\right) = P_0 k e^{kt}$$

$$= k\left(P_0 e^{kt}\right)$$

$$= kP$$

$\Rightarrow$ $P(t) = P_0 e^{kt}$ is a soln to our pop. DE model



exponential growth!

$$P(0) = P_0 e^0 = P_0$$

initial population size

Modifying the population model.

$$\frac{dP}{dt} = kP \quad \text{assumes growth without bound.}$$

Incorporate diminishing growth rate as resources become diminished.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{10000}\right)$$

"carrying capacity" - population limit

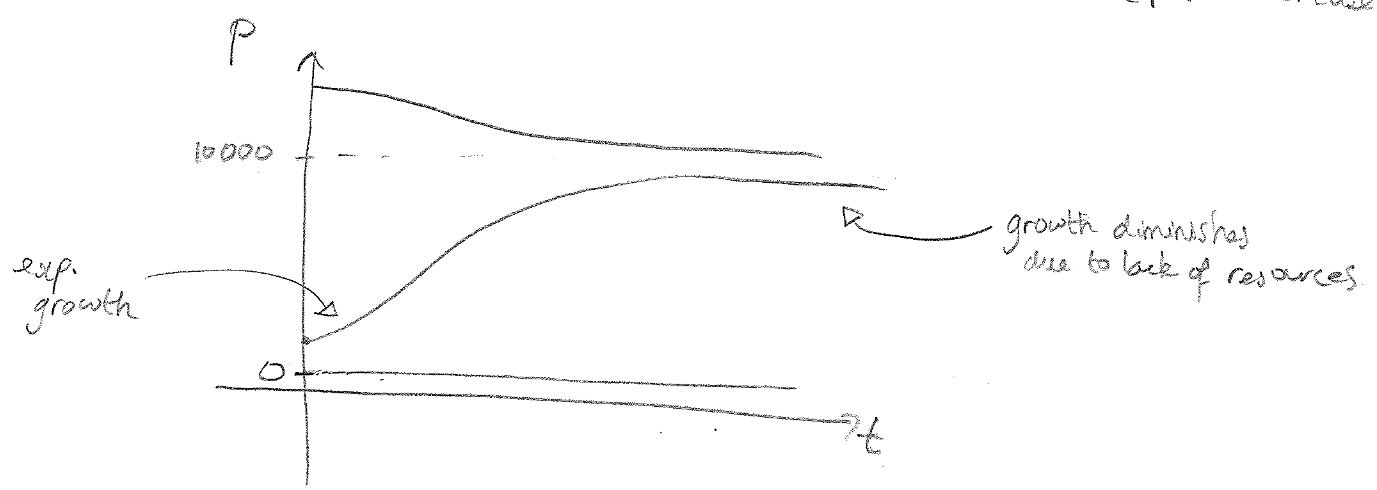
Model behaviour:

for $P \approx 0$, $1 - \frac{P}{10000} \approx 1 \Rightarrow \frac{dP}{dt} \approx kP$ (exp. growth)

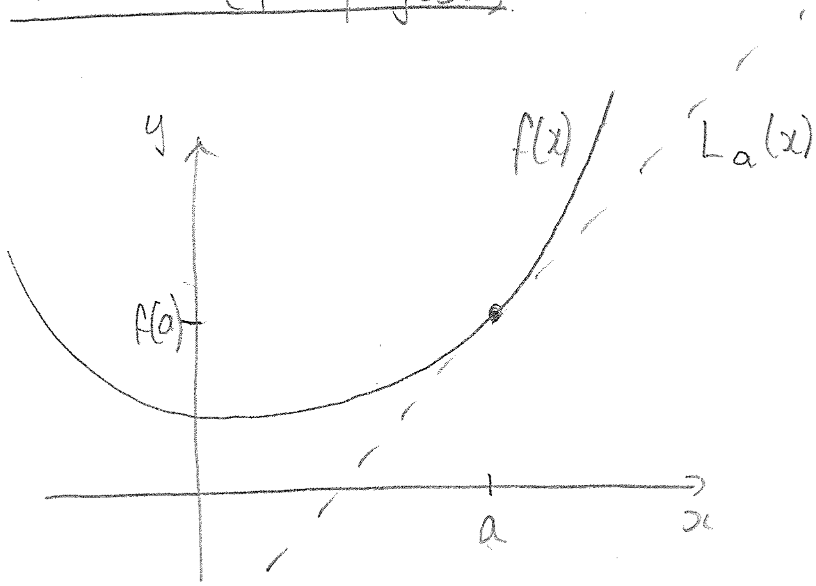
for $P \approx 10\,000$, $1 - \frac{P}{10000} \approx 0 \Rightarrow \frac{dP}{dt} \approx 0$ (growth slows to zero)

for $P \gg 10000$ (over capacity)

$1 - \frac{P}{10000} < 0 \Rightarrow \frac{dP}{dt} < 0$ (pop. decreases)



Aside: (for project)



$L_a(x) \Leftrightarrow$ the <u>linearization</u> of $f(x)$ at the point $x=a$.

$\Leftrightarrow$ tangent line of f at $x=a$ written as a function.

Eqn of straight line through (x_0, y_0) is

$$y - y_0 = m(x - x_0).$$

For tangent line, $m = f'(a)$, $y_0 = f(a)$, $x_0 = a$

$$\Rightarrow \quad y = f(a) + f'(a)(x-a)$$

As a function,

$$L_a(x) = f(a) + f'(a)(x-a).$$

point where line is tangent to $f(x)$.

general $x \in \mathbb{D}$.