Lecture 23.

Today's topics:

- Differentials
- Newton's method

Read Ch 5.4.4

EoL 21 (Newton's method)

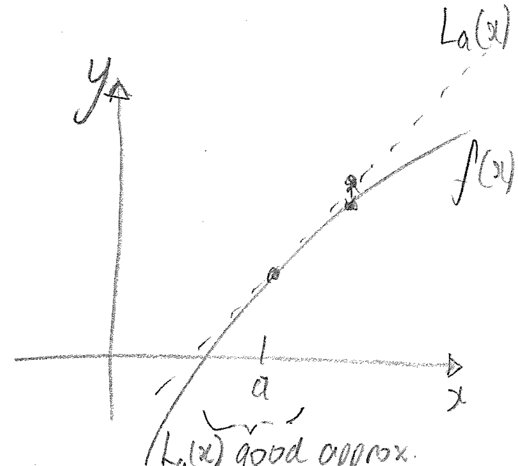
Monday's tutorial
- group work on modelling with exponentials & NM.

Recall:

The tangent line

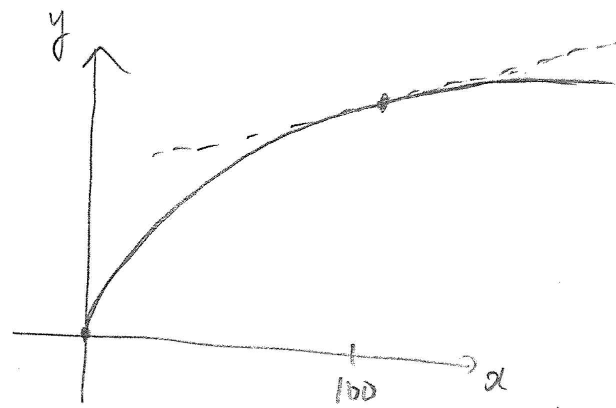
$$L_a(x) = f(a) + f'(a)(x-a)$$

can approximate $f(x)$ for x close to a.



Another example.

Use a linear approximation to estimate $\sqrt{100.5}$.



Approach: let $f(x) = \sqrt{x}$.

$$\Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

Pick 'a' close to 100.5 and nice to evaluate:

$a = 100$, $f(a) = 10$, $f'(a) = \frac{1}{20}$.

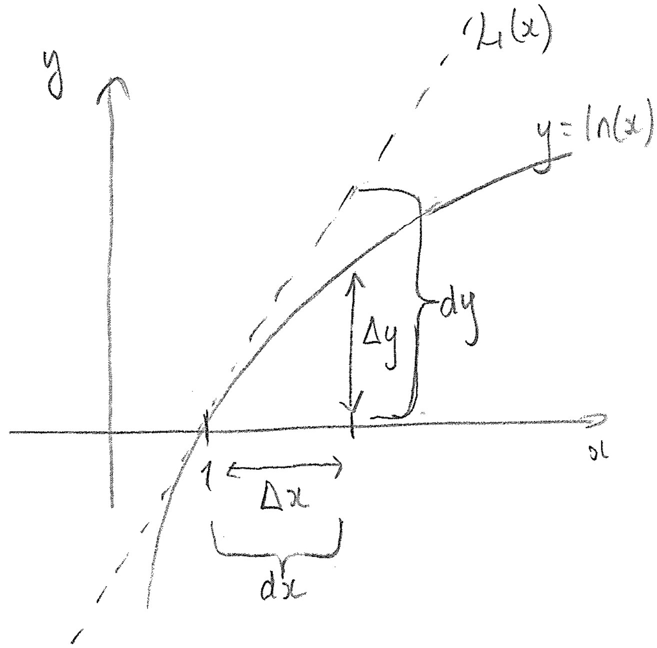
$$L(x) = 10 + \frac{1}{20}(x - 100).$$

$$L(100.5) = 10 + \frac{0.5}{20} = 10.025$$

good approx!

$$\sqrt{100.5} \simeq 10.02497$$

Differentials: "Same concept, new symbols"



- Move Δx ($= dx$) from tangent point:

  Δy: change in $f(x)$ (hard to compute)

  dy: change in $L(x)$ (easy to compute).

- dx, dy referred to as differentials.

Compute dy from dx

We say $dy = f'(x)\,dx$. $\left(\frac{dy}{dx} = f'(x) \right)$

- $f'(x)$: slope at point of linearization
- dx: how far we move from point.

Then $f(a + dx) \simeq f(a) + dy$.

Ex/. Approximate $\sqrt{100.5}$ using differentials.

$$dy = f'(x)\,dx = \frac{1}{2\sqrt{x}}\,dx$$

$$dx = 100.5 - 100 = 0.5$$

$$dy = \frac{1}{2\sqrt{100}}\left(\frac{1}{2}\right) = \frac{1}{40}.$$

$$\Rightarrow f(100 + 0.5) = f(100) + \frac{1}{40} = 10.025.$$

Ex/ (both ways).

Approximate $\frac{1}{4.002}$.

Method 1: Linear approximation

$f(x) = \frac{1}{x}$ Choose $a = 4$

$f'(x) = -\frac{1}{x^2}$

$$L(x) = f(a) + (x-a) f'(a)$$
$$= \frac{1}{4} + (x-4)\left(-\frac{1}{16}\right).$$

$$f(4.002) \simeq L(4.002) = \frac{1}{4} + 0.002\left(-\frac{1}{16}\right) = \frac{1999}{8000}.$$

Method 2: Differentials.

$$dy = f'(x)\,dx$$
$$= -\frac{1}{x^2}(0.002)$$

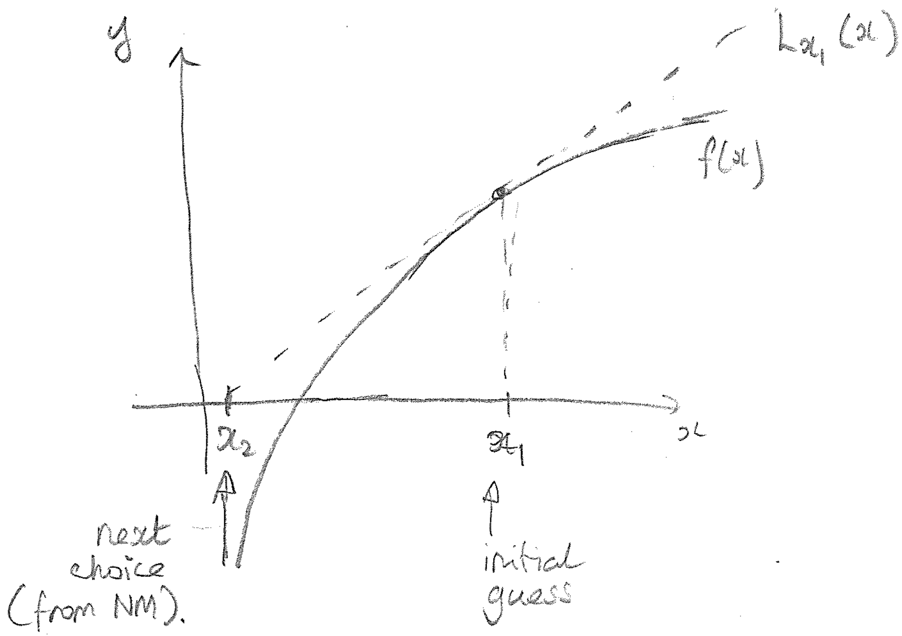
At $x = 4$,

$$dy = -\frac{1}{16}(0.002).$$

$$f(4.002) = f(4) + dy = \frac{1}{4} + 0.002\left(-\frac{1}{16}\right) = \frac{1999}{8000}. \quad \text{(same)}$$

Newton's Method

→ another algorithm to find root of $f(x)=0$.
→ uses tangent lines to approximate f, now with goal of getting closer to the root.



• next guess, x_2 satisfies
$L_{x_1}(x_2)=0$.
(point where tangent line at x_1 crosses axis).

Algebraically.

Recall $L_a(x) = f(a) + (x-a)f'(a)$

$$L_{x_1}(x_2) = 0$$

$$\Rightarrow f(x_1) + (x_2 - x_1)f'(x_1) = 0$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Now we repeat the procedure!

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n \geq 1.$$

Example:

Estimate an x that satisfies

$$\sin(x) = e^x - 2.$$ (root question in disguise).

$$\Rightarrow \underbrace{\sin(x) - e^x + 2}_{f(x)} = 0$$

$f(x)$ ← find root of this function.

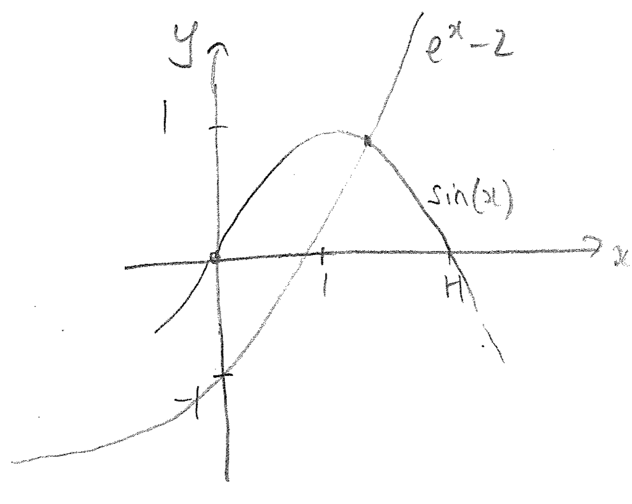
(NM) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

↑ require $f'(x)$

$$f'(x) = \cos(x) - e^x.$$

First guess?

$x_1 = 1$ will do.



$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \frac{\sin(1) - e + 2}{\cos(1) - e}$$

$$= 1.0566$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.0541$$

$x_4 = 1.0541$ $\Rightarrow$ $x = 1.054$ is our approximation

Next time...

- Where does NM fail?
- How can we use NM to approximate e.g. $\sqrt[8]{500}$?