QLSC 600

Bootcamp on nonlinear dynamics

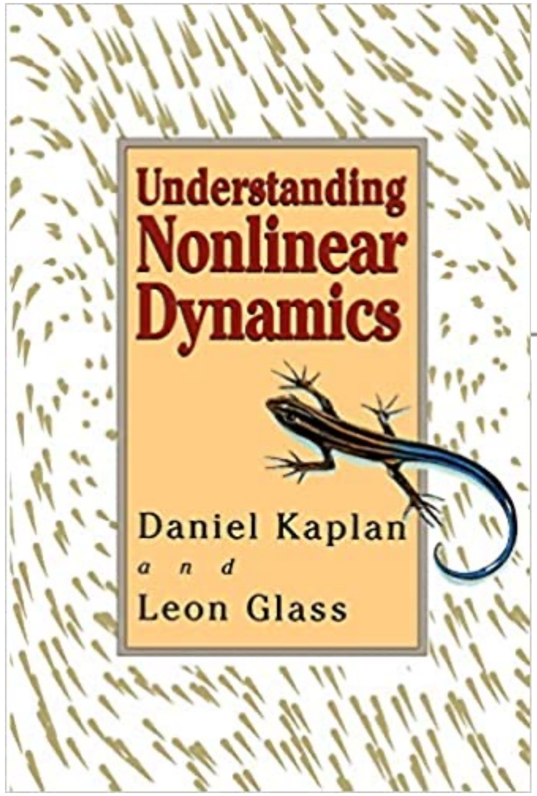
Instructor: Thomas Bury

Supplementary reading

McGill | McGill Library

https://www.mcgill.ca/library/

View eBook



Chapters

1 FINITE-DIFFERENCE EQUATIONS

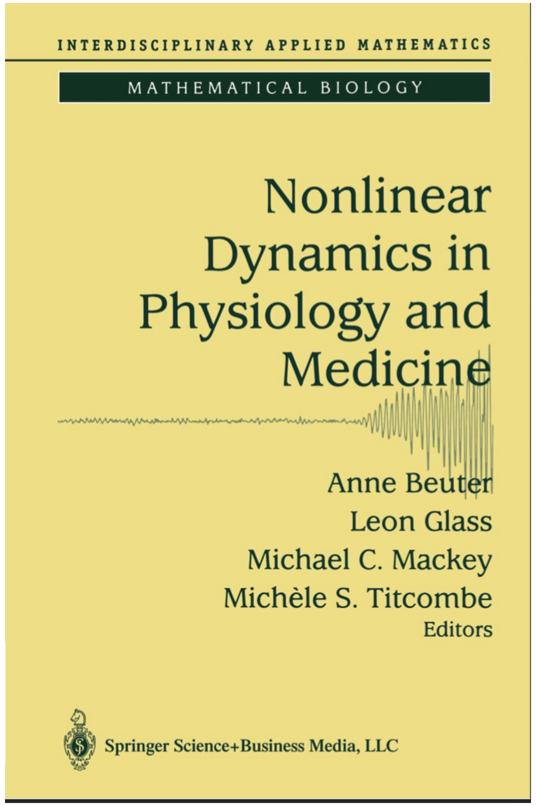
4 ONE-DIMENSIONAL DIFFERENTIAL EQUATIONS

5 TWO-DIMENSIONAL DIFFERENTIAL EQUATIONS

Chapters

1 Theoretical Approaches in Physiology
Michael C. Mackey and Anne Beuter

5 Resetting and Entraining Biological Rhythms
Leon Glass

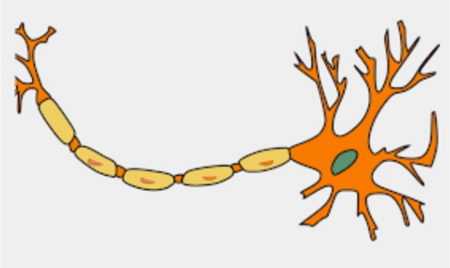


A few biological applications…

Infections disease modelling[1]

$$\frac{dS}{dt} = -\frac{\beta I S}{N}, \quad \frac{dI}{dt} = \frac{\beta I S}{N} - \gamma I, \quad \frac{dR}{dt} = \gamma I.$$

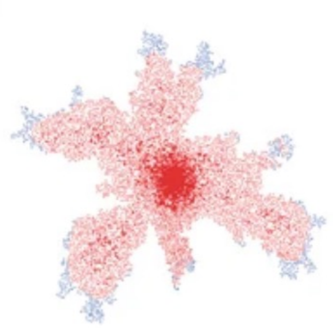
Physiology and neuroscience[3]



$$\frac{dV}{dt} = V(V-a)(V-1) - W - I_{app}$$

$$\frac{dW}{dt} = \epsilon(V - \gamma W)$$

Oncology[2]

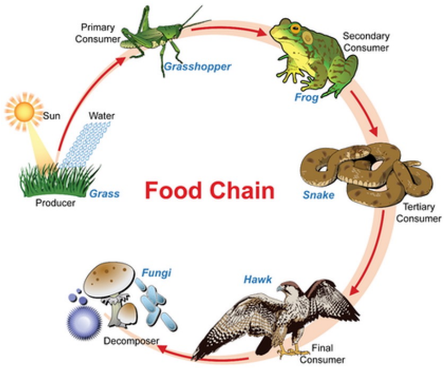


$$\frac{\delta m}{\delta t} = D_m \nabla^2 m + \mu n_{ij} - \lambda m,$$

$$\frac{\delta f}{\delta t} = -\delta m f,$$

$$\frac{\delta c}{\delta t} = D_c \nabla^2 m + \beta f - \gamma n_{ij} c - \alpha c.$$

Ecology and evolution[4]



$$x_{n+1} = r x_n (1 - x_n)$$

[1]Anderson, Roy M., and Robert M. May. *Infectious diseases of humans: dynamics and control*. Oxford university press, 1992.

[2]Anderson, Alexander RA, and Vito Quaranta. "Integrative mathematical oncology." *Nature Reviews Cancer* 8.3 (2008): 227-234.

[3]Glass, Leon, and Michael C. Mackey. *From clocks to chaos*. Princeton University Press, 2020.

[4]Levin, Simon A., Thomas G. Hallam, and Louis J. Gross, eds. *Applied mathematical ecology*. Vol. 18. Springer Science & Business Media, 2012

Agenda

Part I: Difference equations

- Linear difference equations
  - Algebraic solutions
  - Cobweb diagrams
- Nonlinear difference equations
  - Logistic map
  - Bifurcations and chaos

$$x_{t+1} = f(x_t)$$

Part II: Differential equations

- Comparison with difference equations
- Numerical analysis

$$\frac{dx}{dt} = f(x)$$

Difference equation

$$x_{t+1} = f(x_t)$$

x_t Number of flies caught the summer of year t

(Depends mostly on number of eggs laid in the previous year)

(Kaplan & Glass)

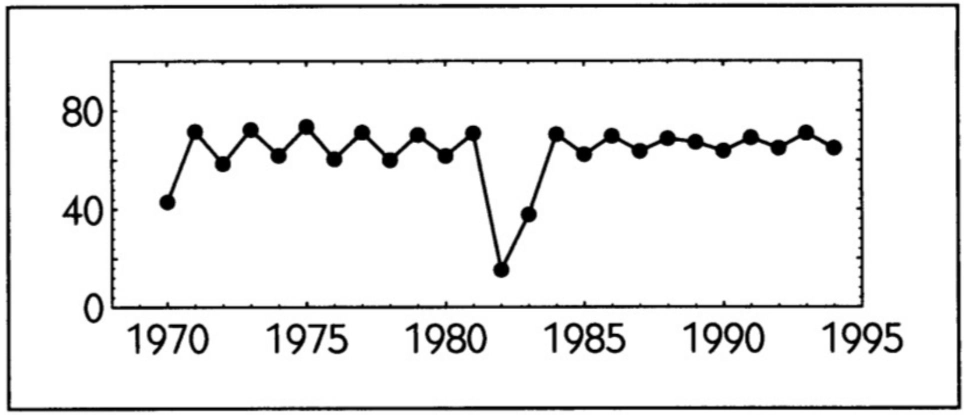


Figure 1.1 The number of flies caught during the annual fly survey.

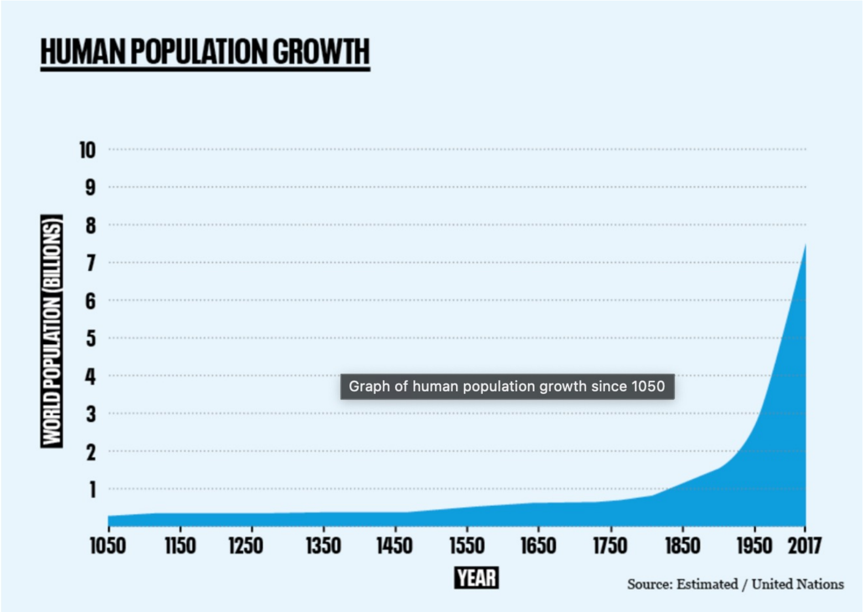
vs.

Differential equation

$$\frac{dx}{dt} = f(x)$$

$x(t)$ World population at time t

(Depends on continuous birth and death rates)



www.populationmatters.org/population-numbers

$$x_{t+1} = f(x_t)$$

Part I

Difference equations

Linear difference equation

$$x_{t+1} = r x_t$$

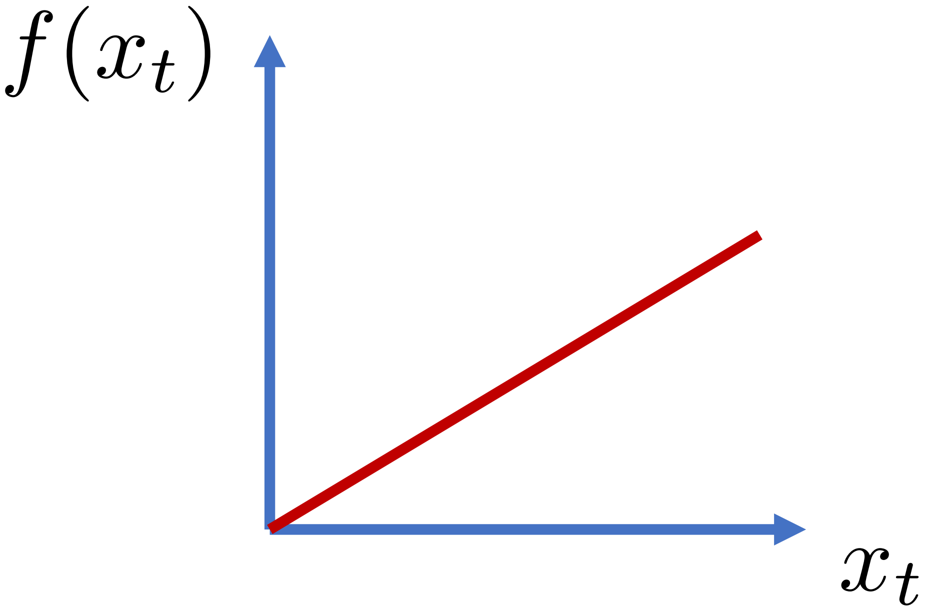
How can we solve this?

Let's say we have 100 flies at t=0

$$x_0 = 100$$

Then...

$$x_1 = r x_0 = 100r$$
$$x_2 = r x_1 = 100r^2$$
$$x_3 = r x_2 = 100r^3$$
$$\vdots$$



Solution:

$$x_t = r^t x_0$$

Case r > 1

$$x_{t+1} = r x_t$$

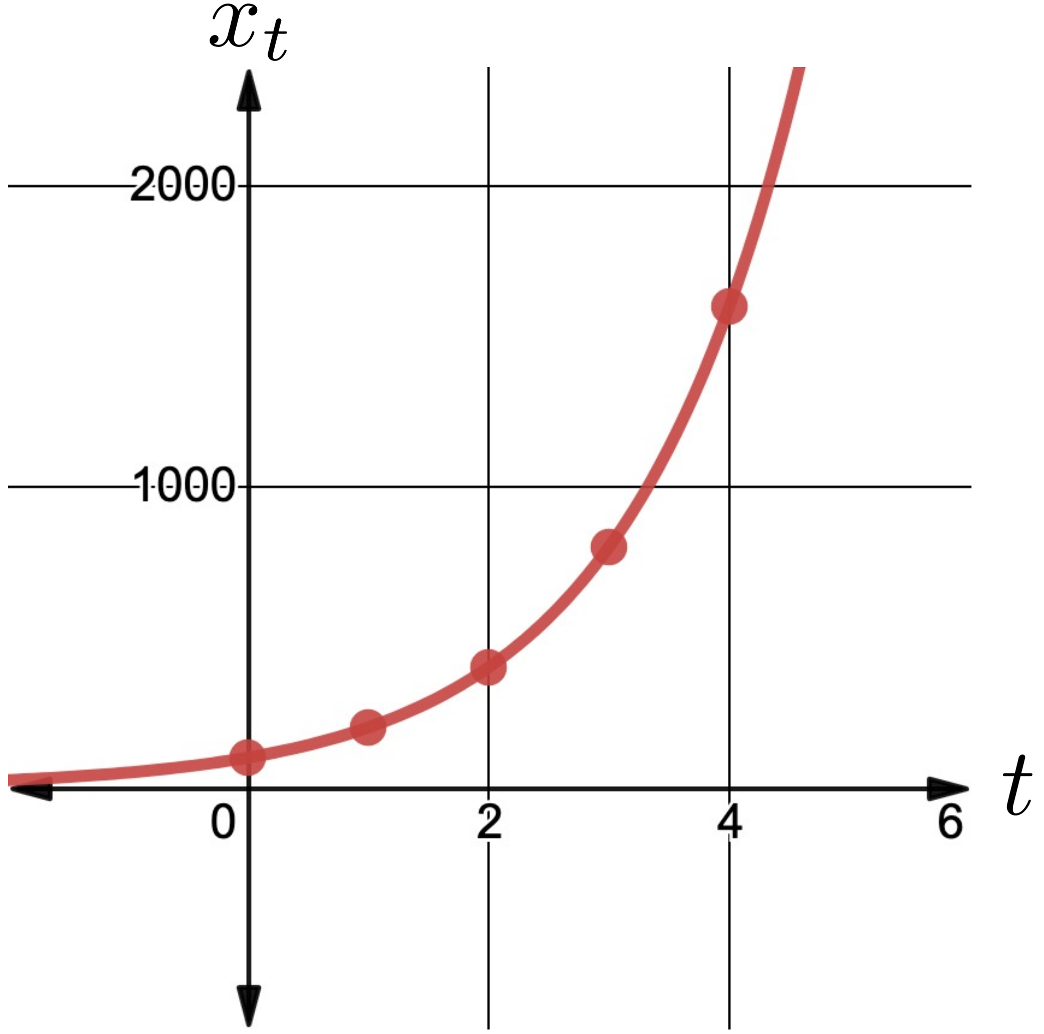
$$x_t = r^t x_0$$

Exponential growth

as $t \to \infty$?

$$x_t \to \infty$$

$$x_t = 100.2^t$$

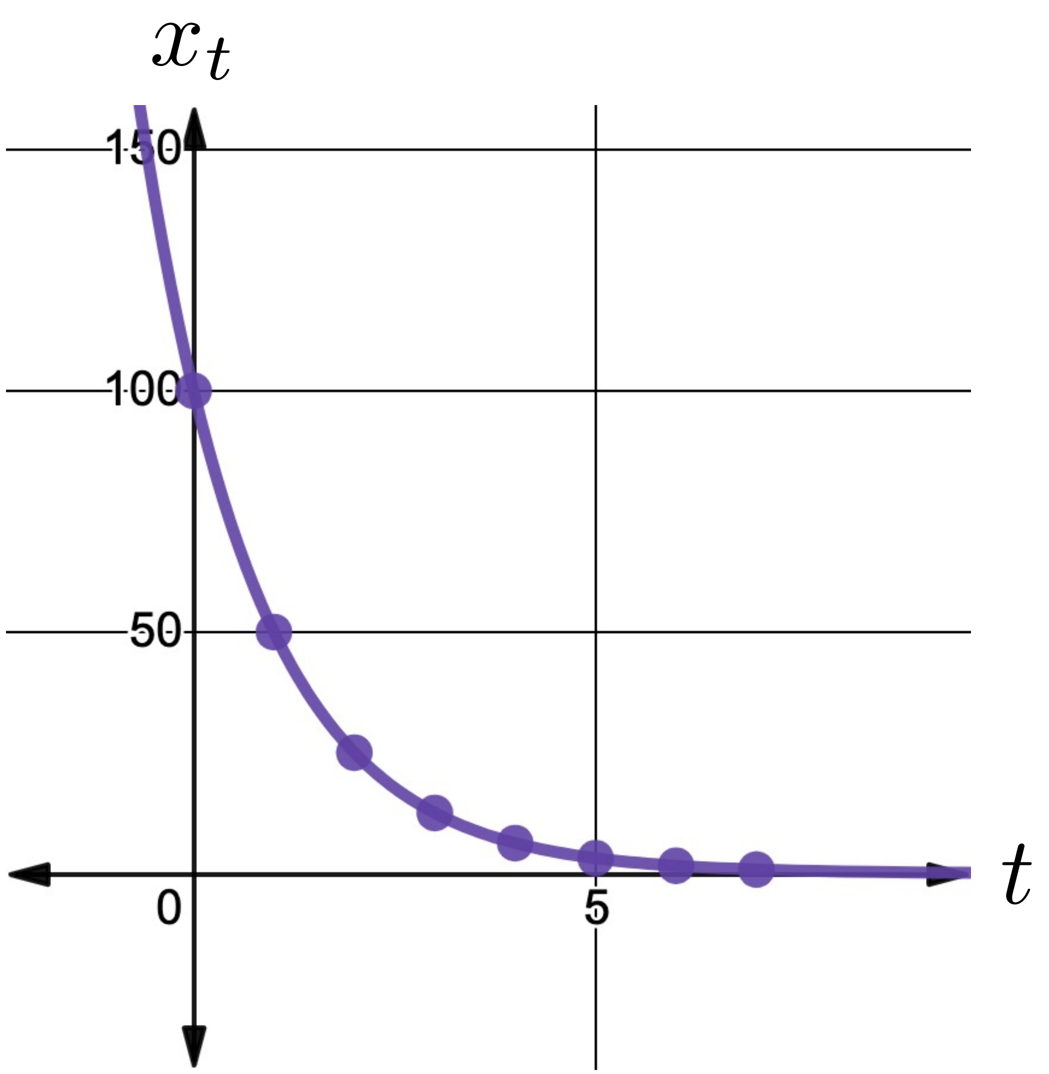


Case 0< r < 1

$$x_t = 100.(0.5^t)$$

$$x_{t+1} = r x_t$$

$$x_t = r^t x_0$$



Exponential decay

as $t \to \infty$?

$$x_t \to 0$$

Case r=1

$$x_{t+1} = r x_t$$

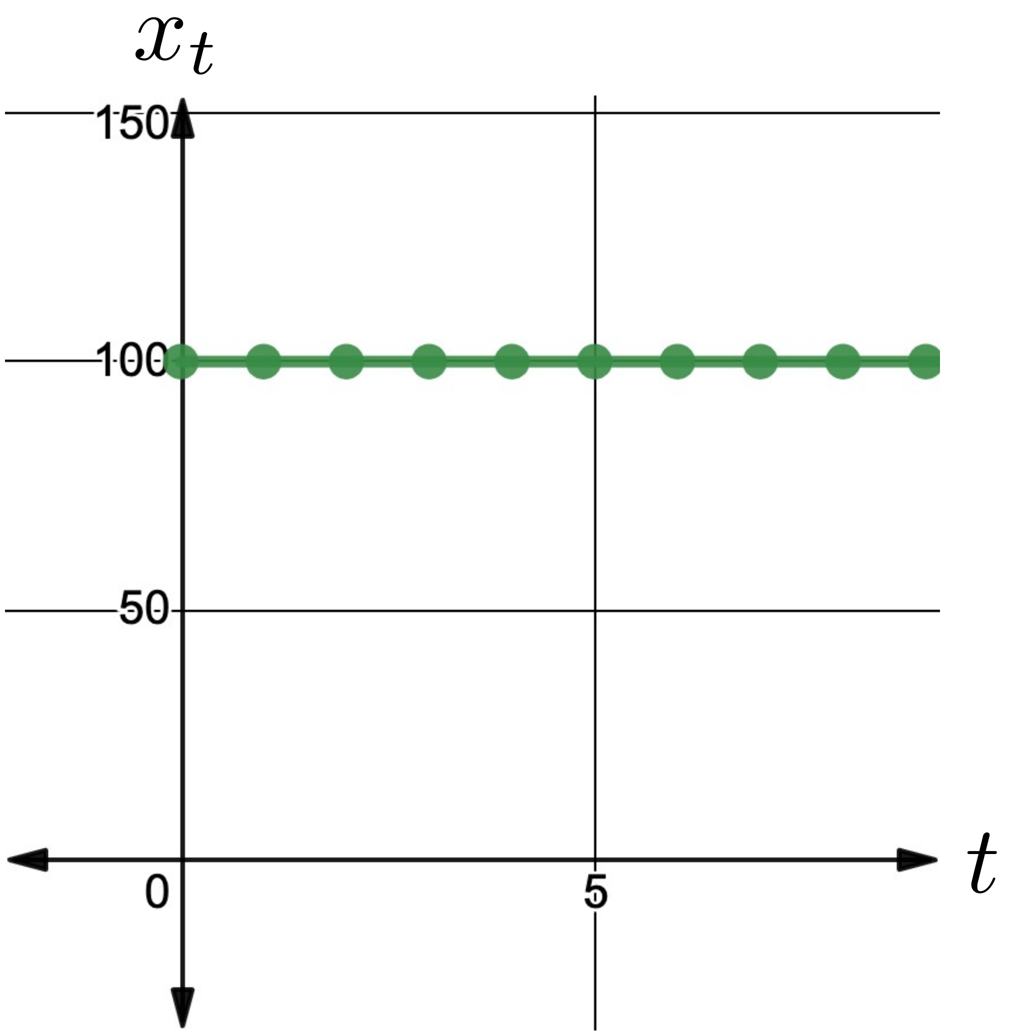
$$x_t = r^t x_0$$

Fixed point / Steady state

as $t \to \infty$?

$$x_t \to x_0$$

$$x_t = 100.(1^t)$$



Case -1< r < 0

$$x_{t+1} = rx_t$$

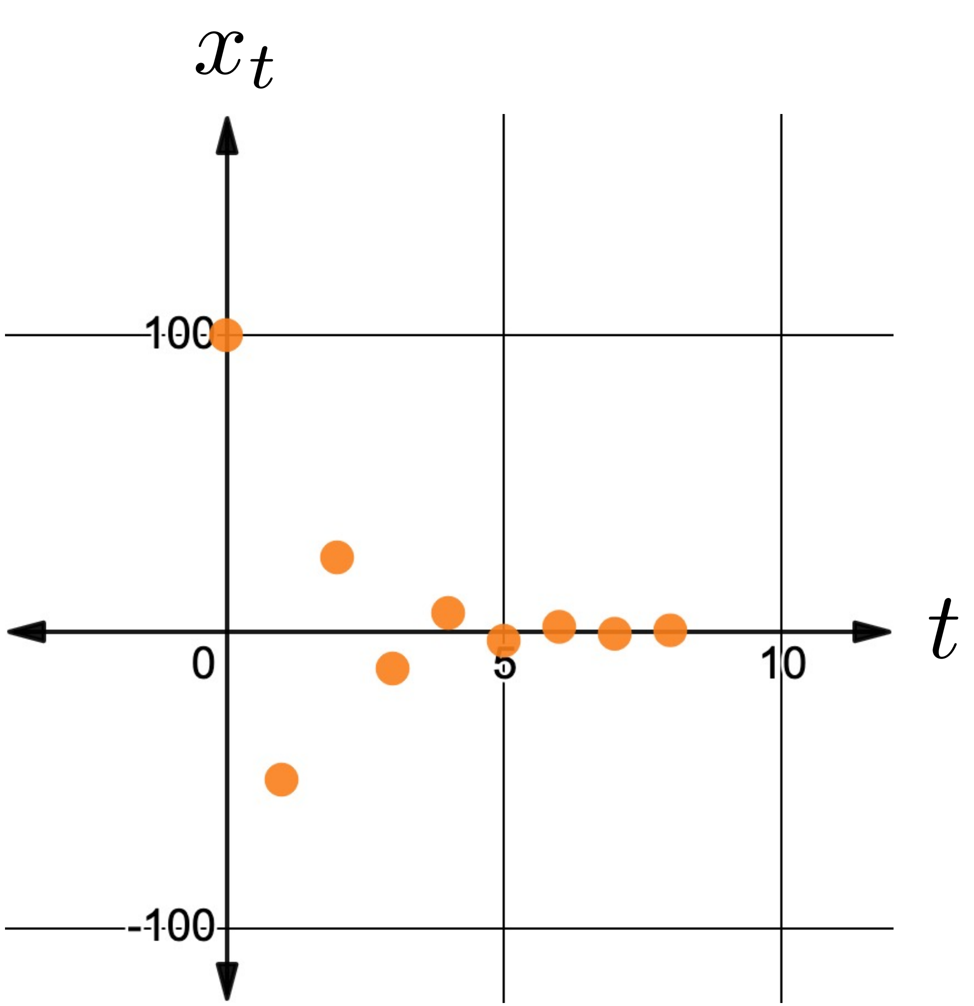
$$x_t = r^t x_0$$

Oscillatory decay

as $t \to \infty$?

$$x_t \to 0$$

$$x_t = 100.(-0.5^t)$$



Case r < -1

$$x_{t+1} = rx_t$$

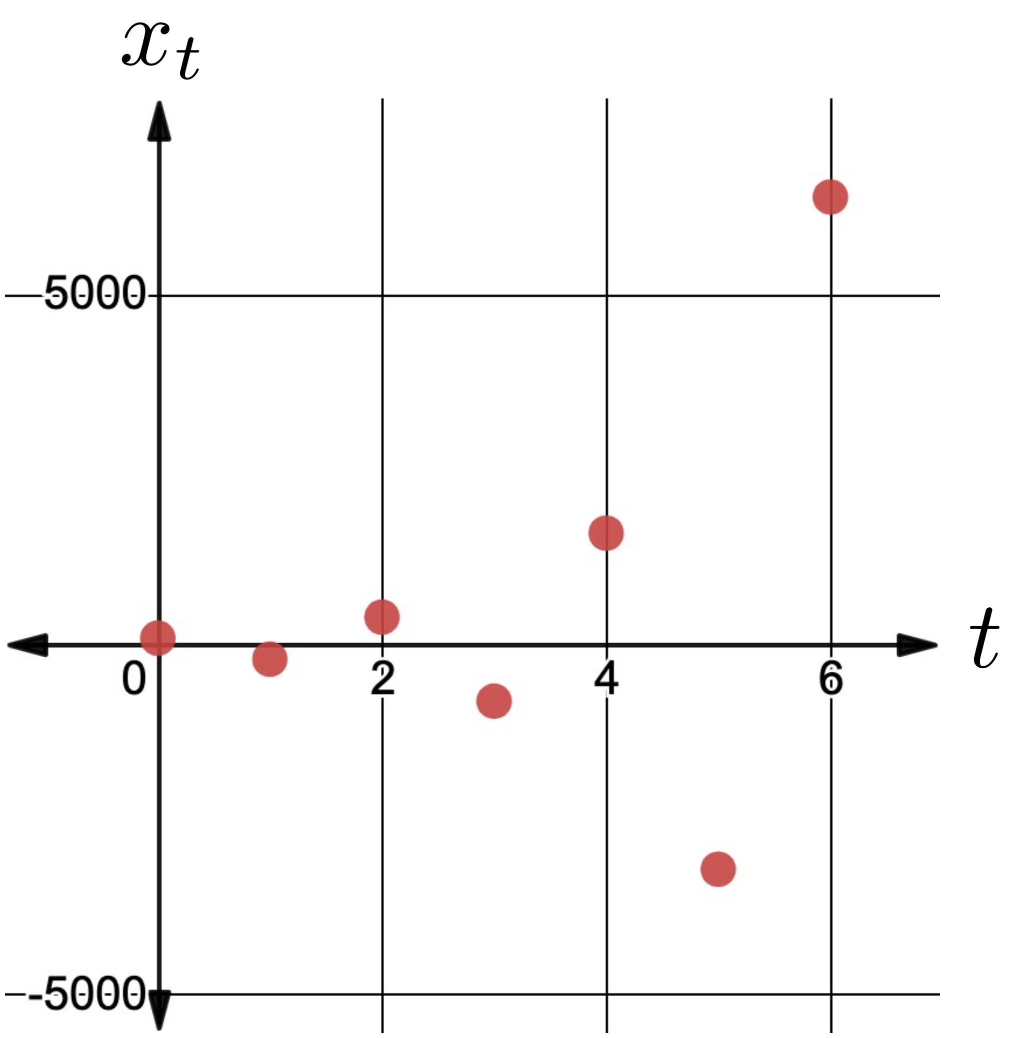
$$x_t = r^t x_0$$

Oscillatory growth

as $t \to \infty$?

$$x_t \to \pm\infty$$

$$x_t = 100.(-2)^t$$



Case r = -1

$$x_{t+1} = r x_t$$

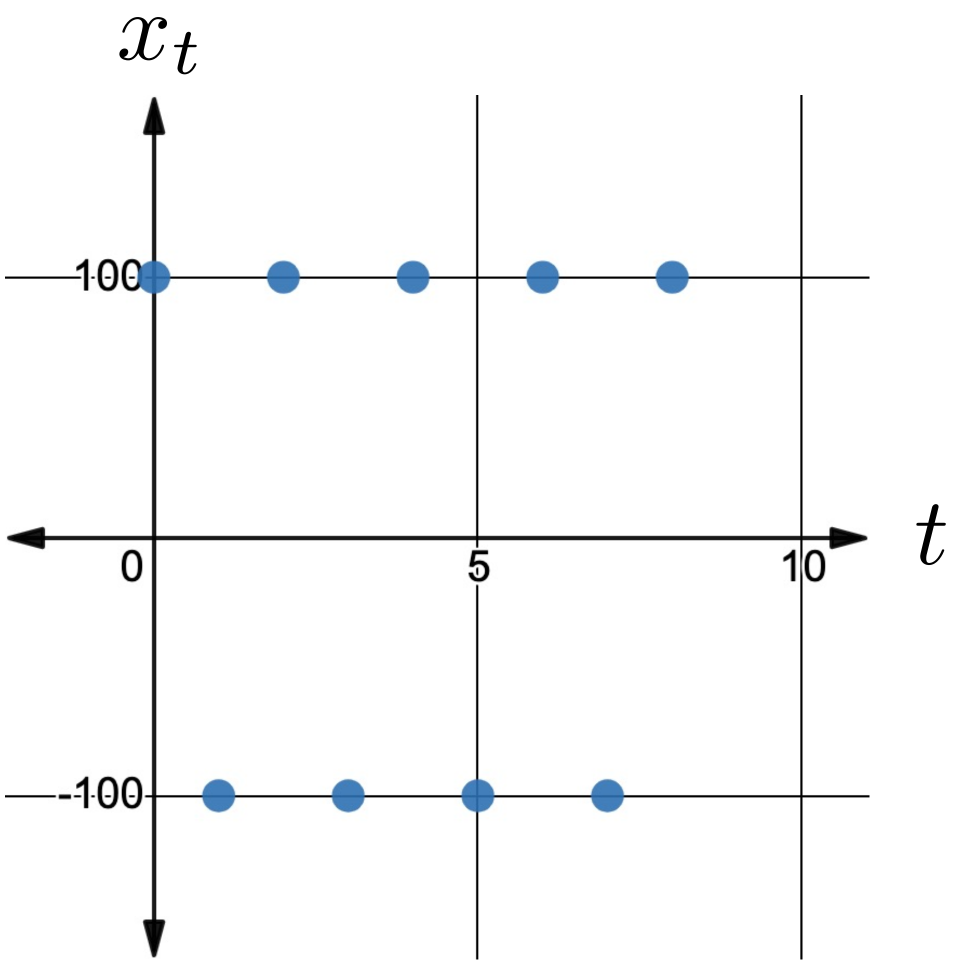
$$x_t = r^t x_0$$

Periodic cycle
(period 2)

as $t \to \infty$?

$$x_t \to \pm x_0$$

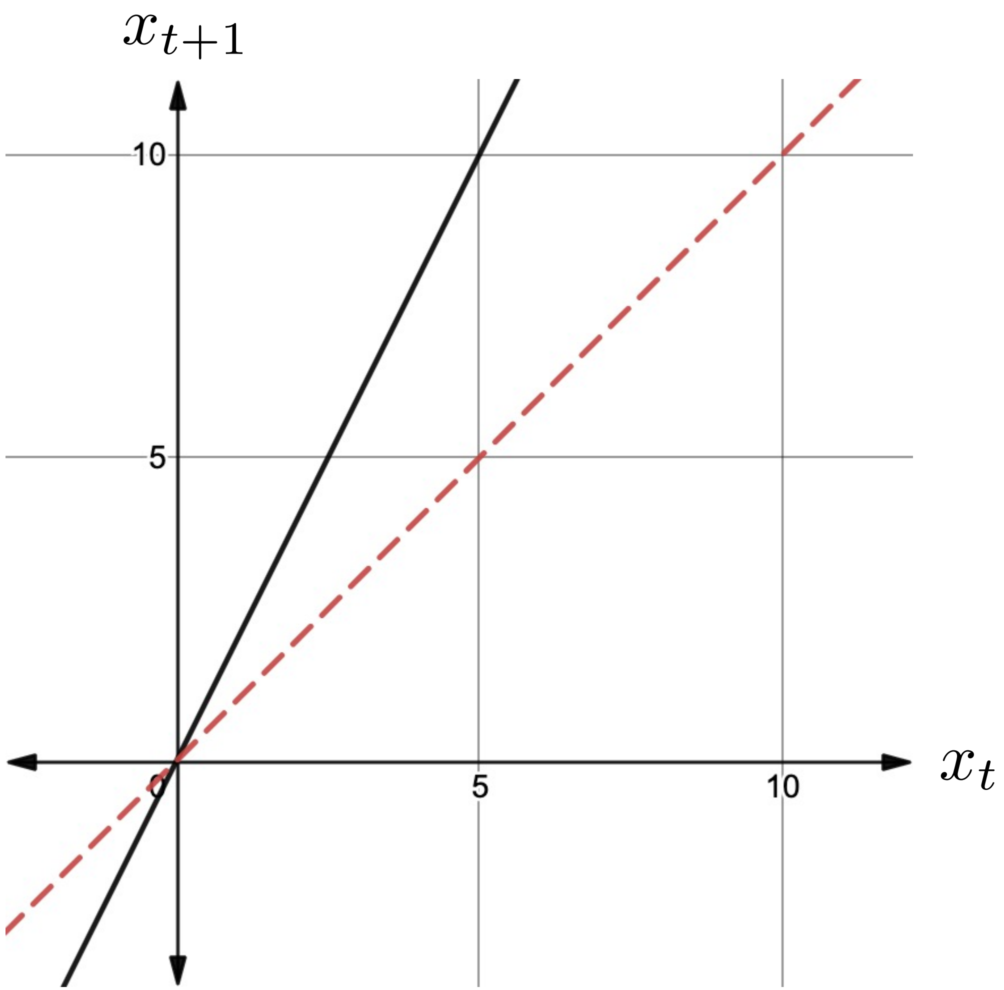
$$x_t = 100.(-1)^t$$



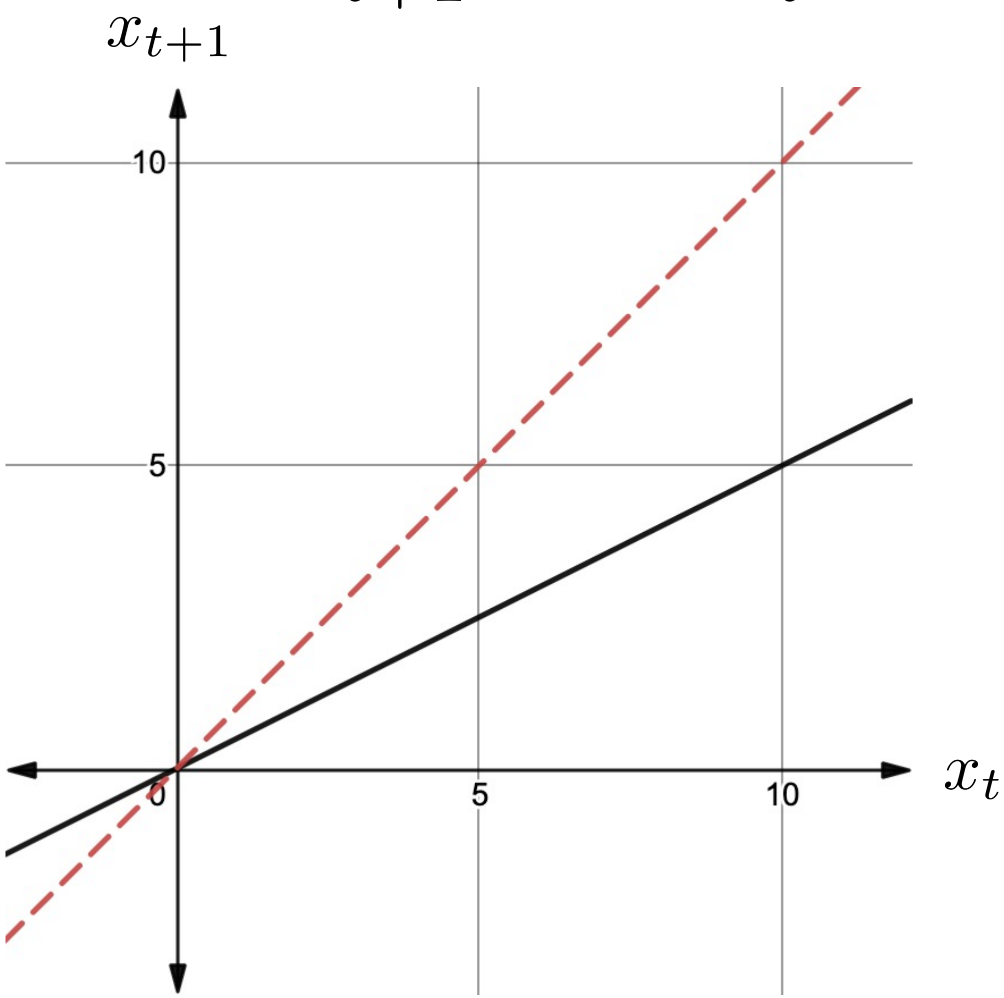
Graphical iteration – cobweb plots

Analytic solution can intractable – graphical iteration provides a quick qualitative description

$$x_{t+1} = 2x_t$$

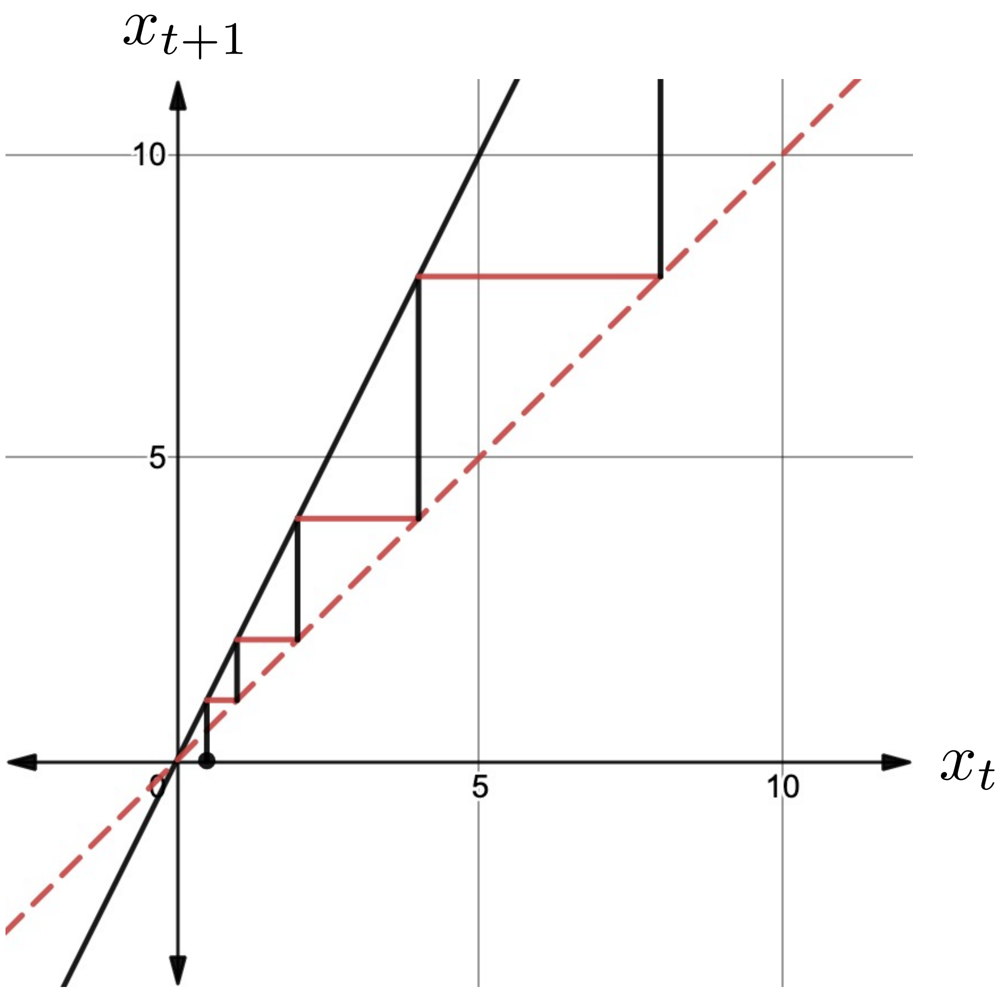


$$x_{t+1} = 0.5x_t$$

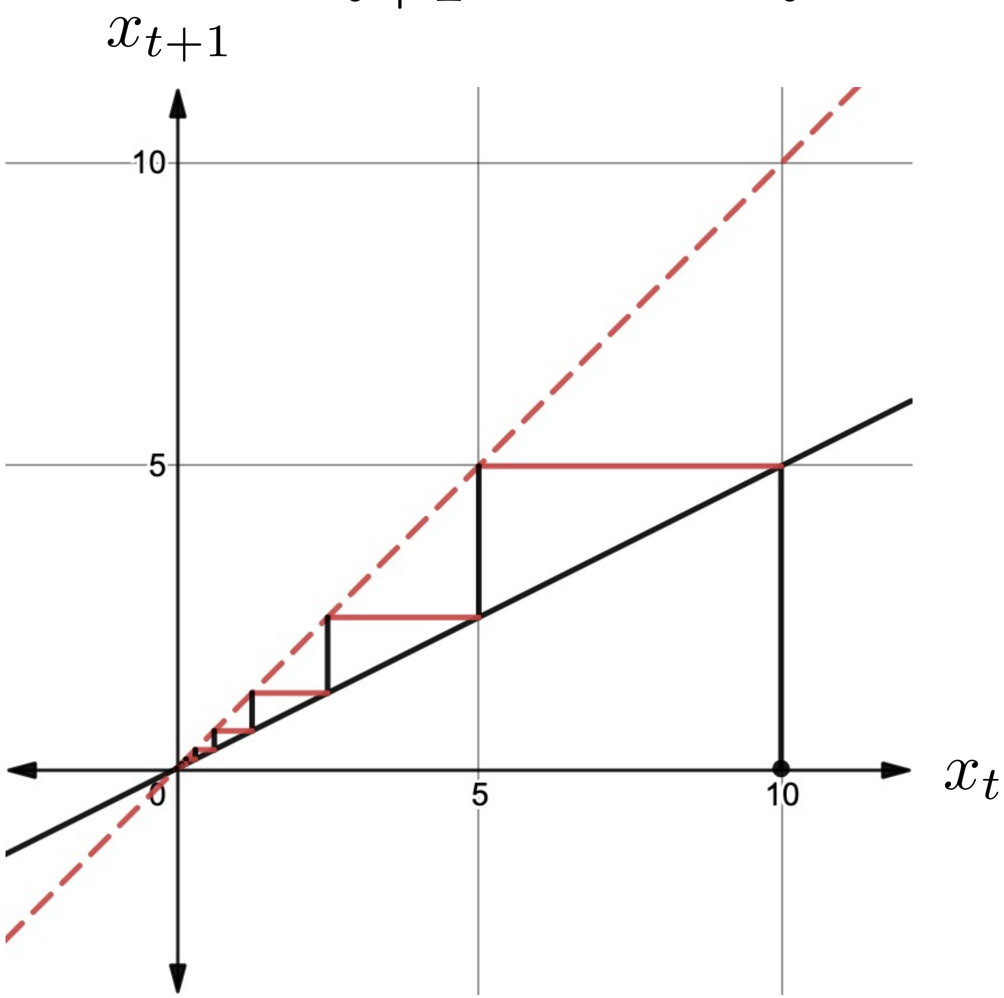


Graphical iteration – cobweb plots

$$x_{t+1} = 2x_t$$



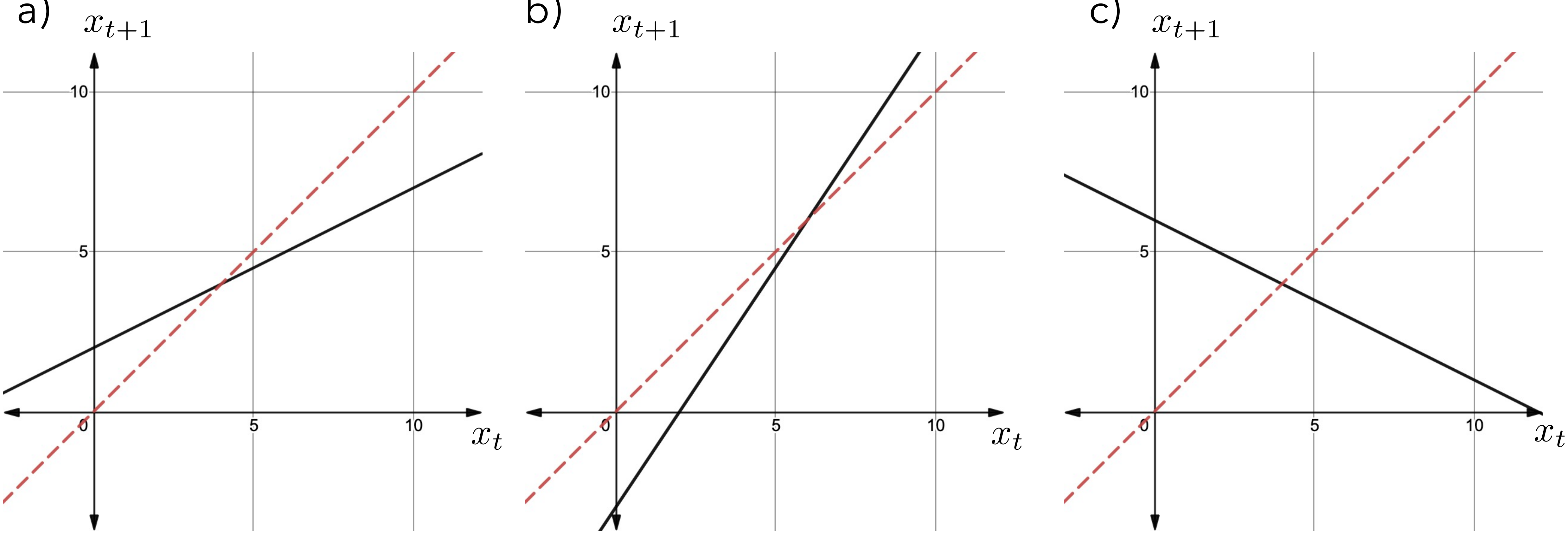
$$x_{t+1} = 0.5x_t$$



Try these…

Describe the qualitative behaviour that you observe

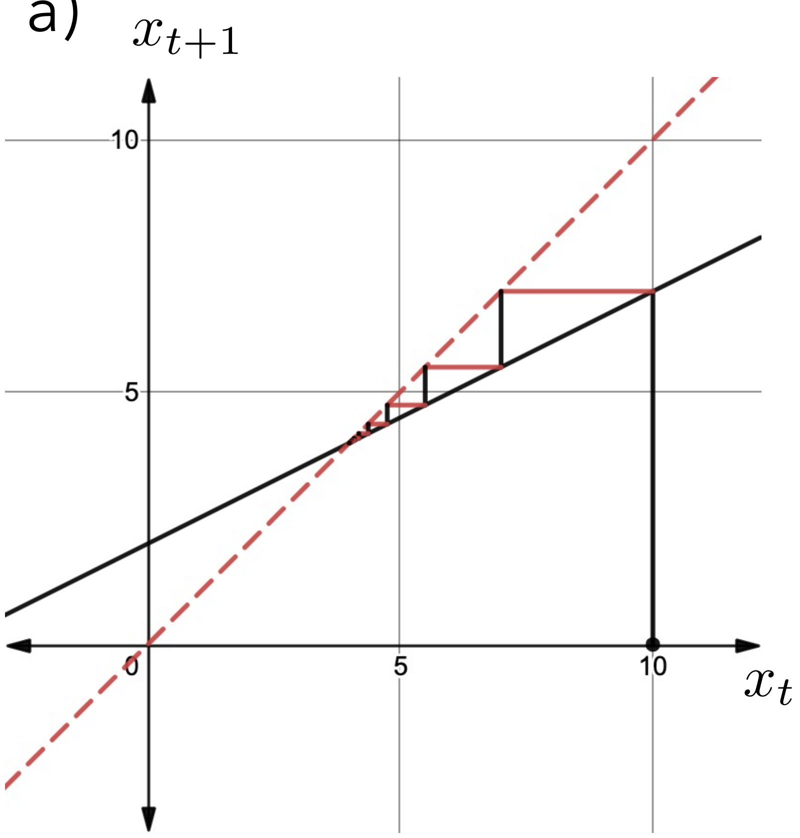
a) b) c)



Try these…

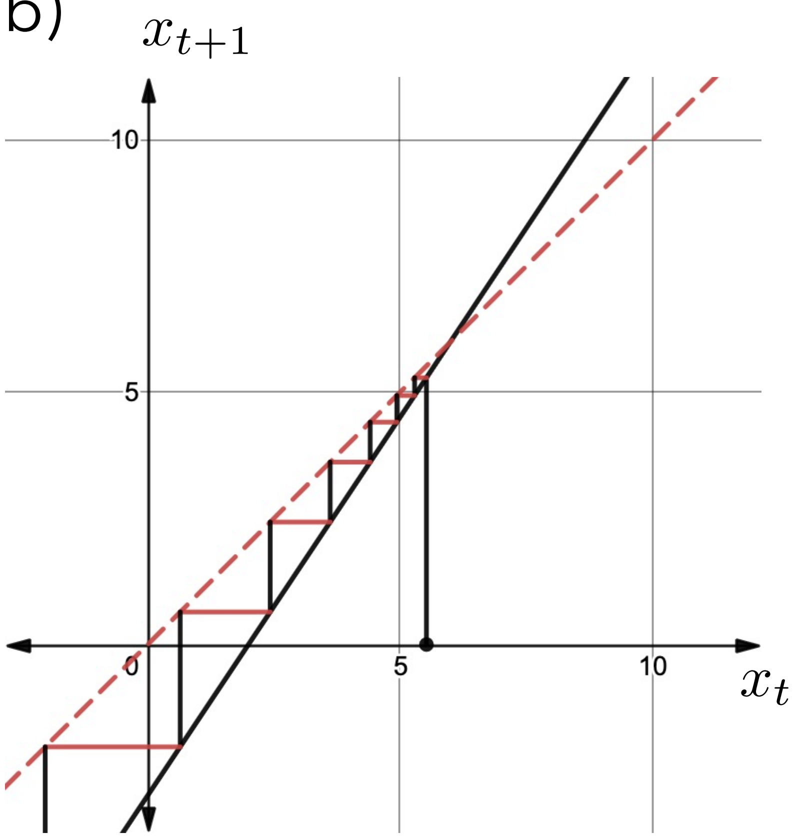
Describe the qualitative behaviour that you observe

a)



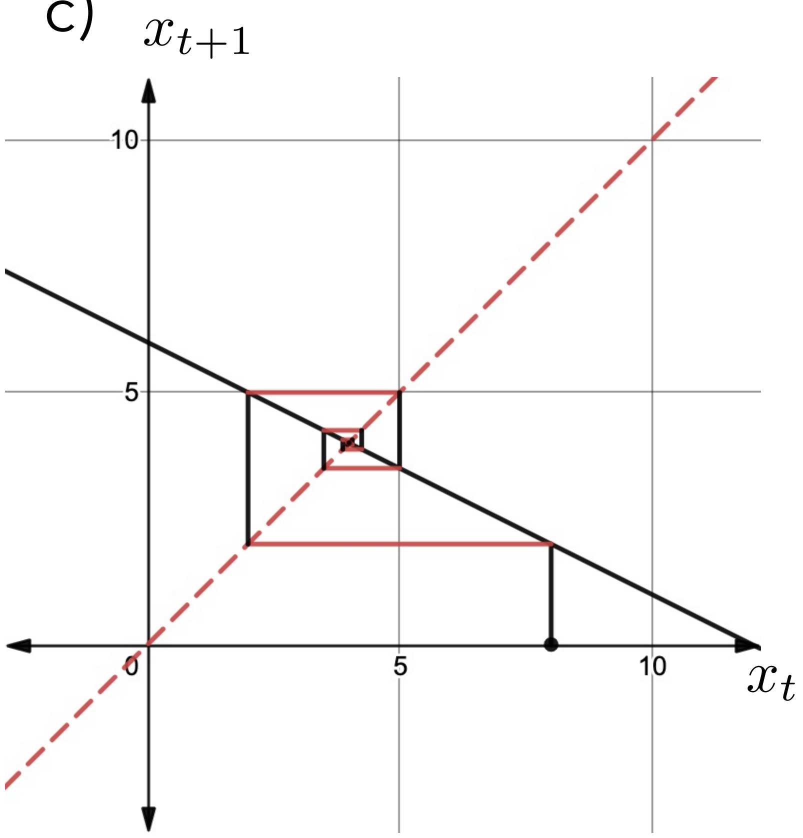
Stable fixed point with
monotonic decay

b)



Unstable fixed point
with monotonic growth

c)



Stable fixed point with
oscillatory decay

Nonlinear difference equations

(Kaplan & Glass)

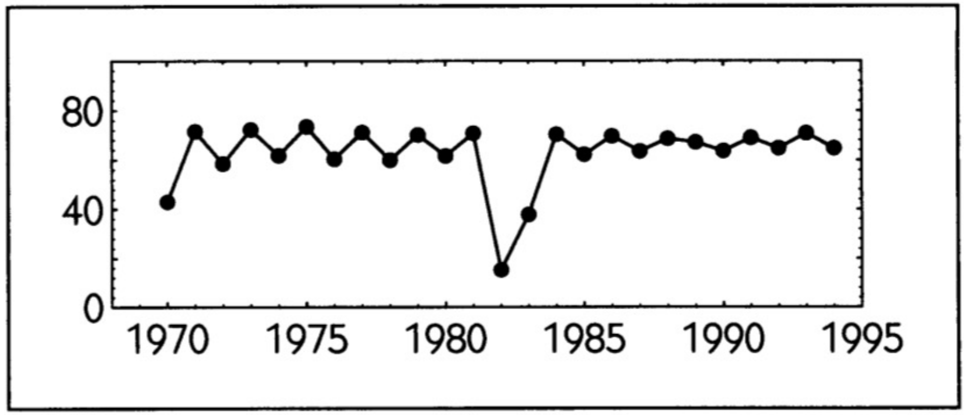


Figure 1.1 The number of flies caught during the annual fly survey.

Is the linear difference equation

$$N_{t+1} = rN_t$$

a good model for population dynamics?

Assumes growth rate per capita is fixed

However, as N_t gets big, resources become limited…

$$N_{t+1} = (r - bN_t)N_t$$

Nonlinear

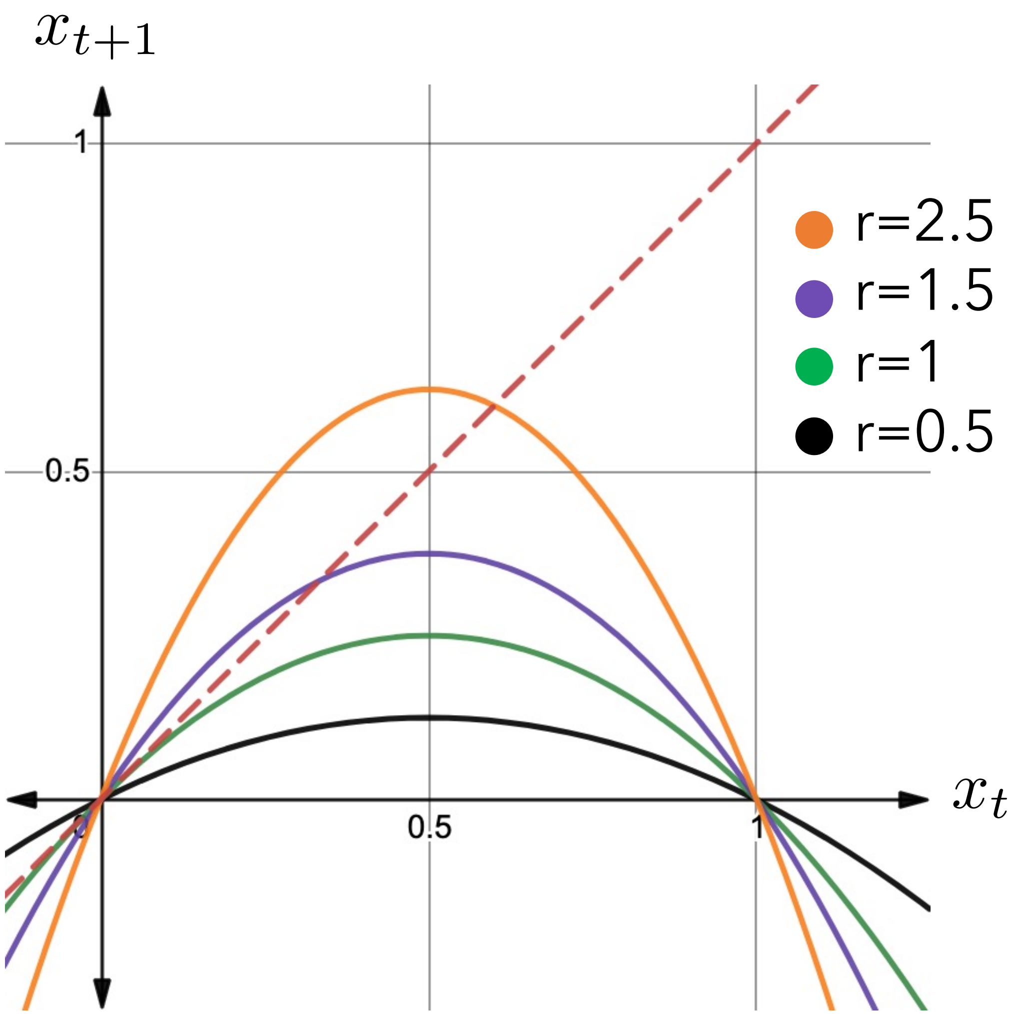
Simplify with substitution $x_t = \frac{b}{R} N_t$

Gives $$x_{t+1} = r(1 - x_t)x_t$$

Logistic map

The logistic map

$$x_{t+1} = r(1 - x_t)x_t$$



Dynamics vary as a function of r

At what values of r do the dynamics qualitatively change (bifurcations)?

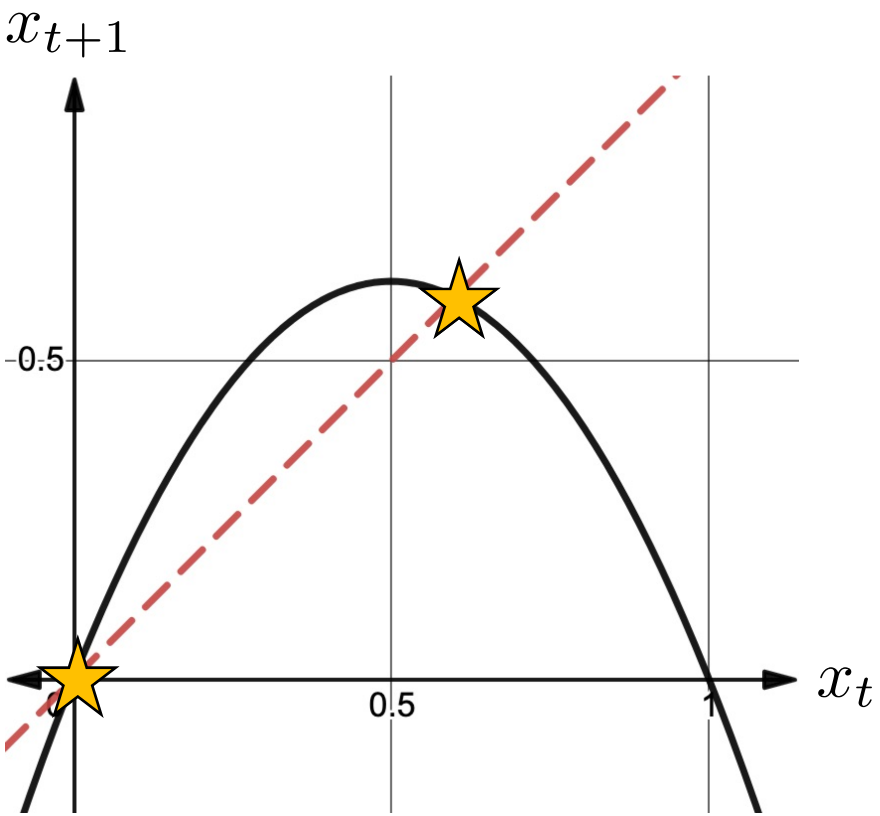
Two important features of nonlinear dynamics:
equilibria and stability

Equilibria of a difference equation $x_{t+1} = f(x_t)$

x^* is an equilibrium of the difference equation if and only if

$$x^* = f(x^*)$$

(intersection of lines y=f(x) and y=x)



E.g. logistic map:

$$x^* = rx^*(1 - x^*)$$

Quadratic equation with roots:

$$x^* = 0 \qquad x^* = \frac{r-1}{r}$$

Latter is biologically plausible for

$$r > 1$$

Stability of equilibria

x^* is a stable equilibrium if

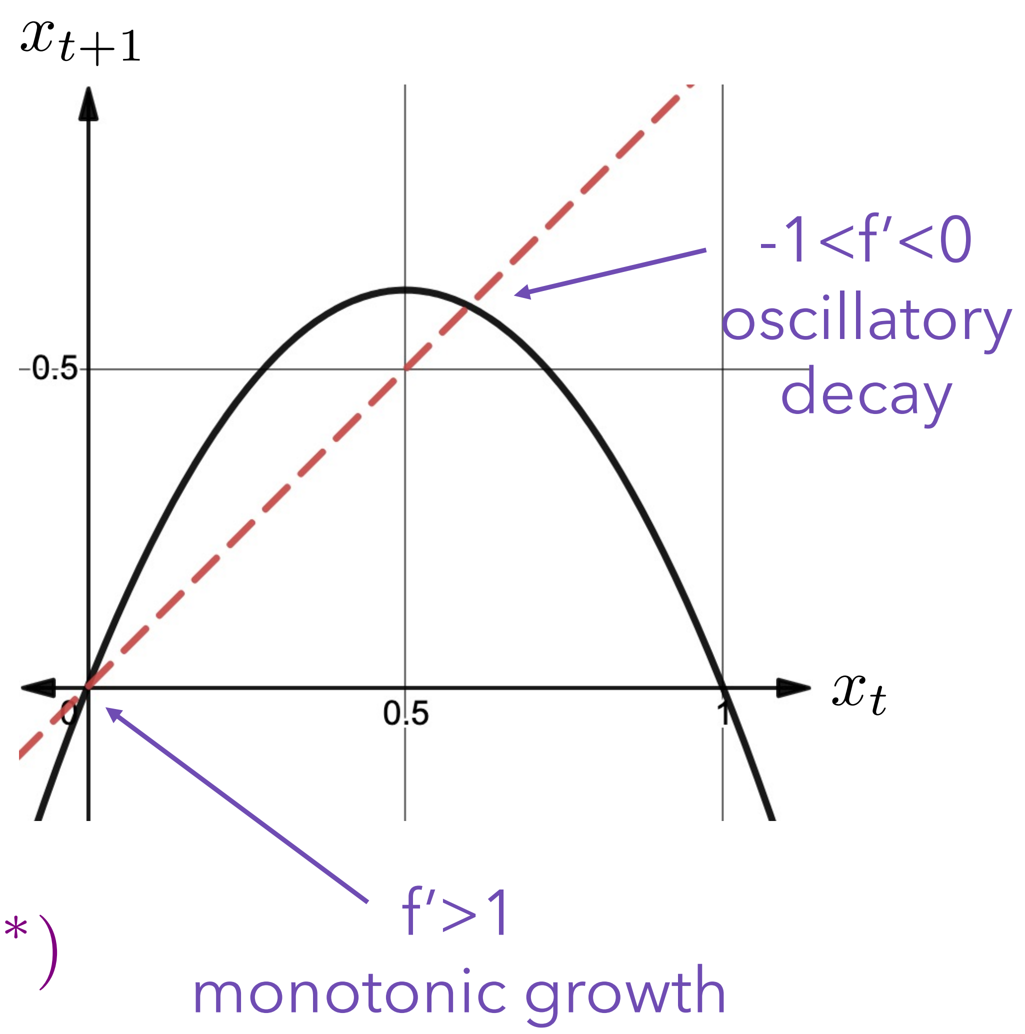
$$|f'(x^*)| < 1$$

and unstable if

$$|f'(x^*)| > 1$$

$|f'(x^*)| = 1$ is inconclusive

(gradient of y=f(x) at intersection with y=x)



Linear difference equation

$$\epsilon_{t+1} = r\epsilon_t$$

describes the nearby dynamics

$r = f'(x^*)$

MATLAB exercise

Investigate the dynamics of the logistic map at different values of r

$$x_{t+1} = r(1 - x_t)x_t$$

Code: https://cnd.mcgill.ca/ebook/index.html or *myCourses*

`quadmap.m`

`fditer.m`

`cobweb.m`

`bifurc.m`

Iterate the logistic map N times

```
y=fditer('quadmap', x0, r, N)
plot(y,'+')
```

Make a cobweb plot for N iterations

```
cobweb('quadmap', x0, r, N)
```

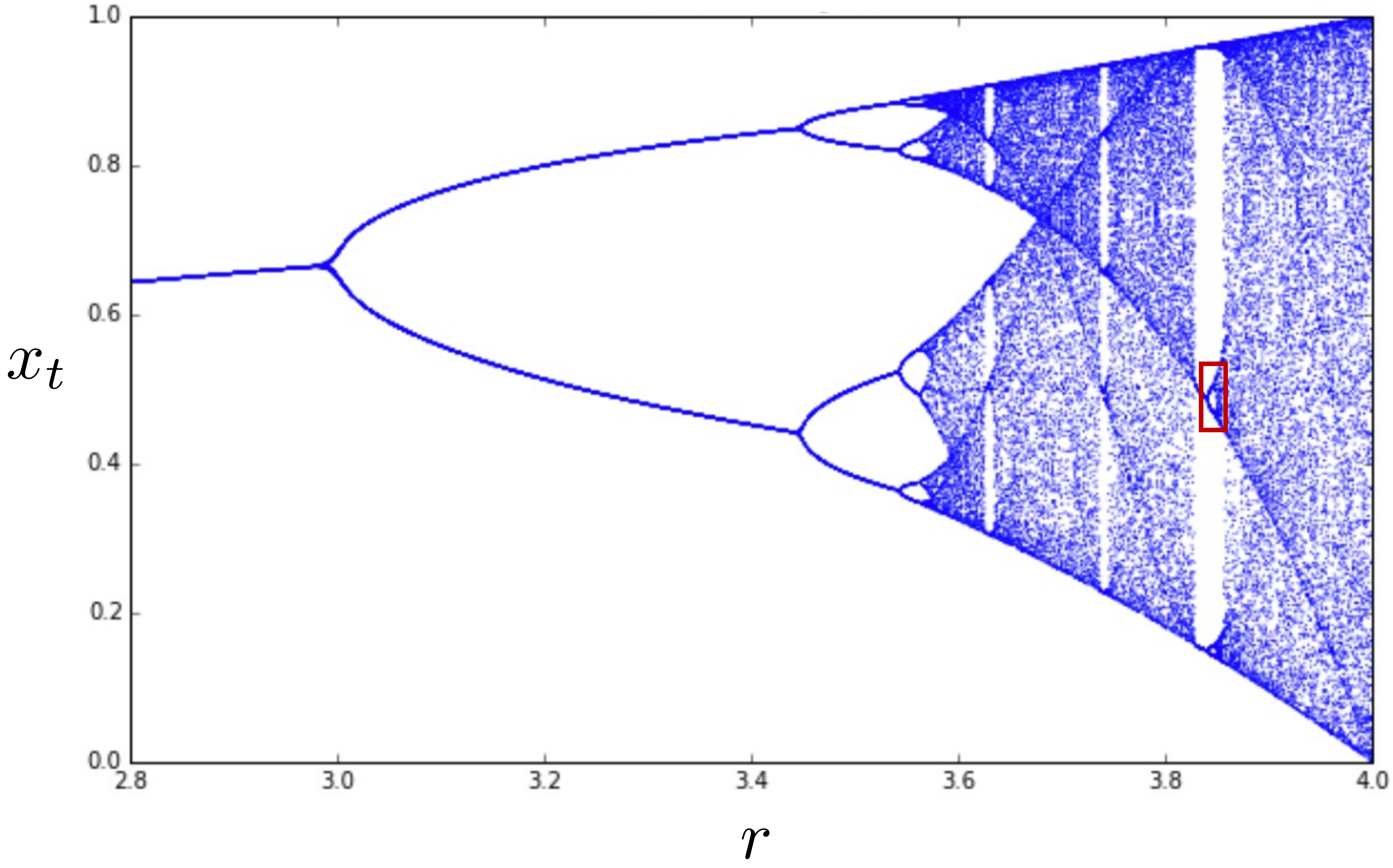
Plot long-term dynamics as a function of r (bifurcation diagram)

```
bifurc('quadmap', rmin, rmax)
```

Chaos

- Bounded
- Aperiodic
- Sensitive to initial conditions
- Deterministic

Bifurcation diagram



https://geoffboeing.com/2015/03/chaos-theory-logistic-map/

Fractal structure (self-similar)

One-to-one map with
Mandelbrot set:

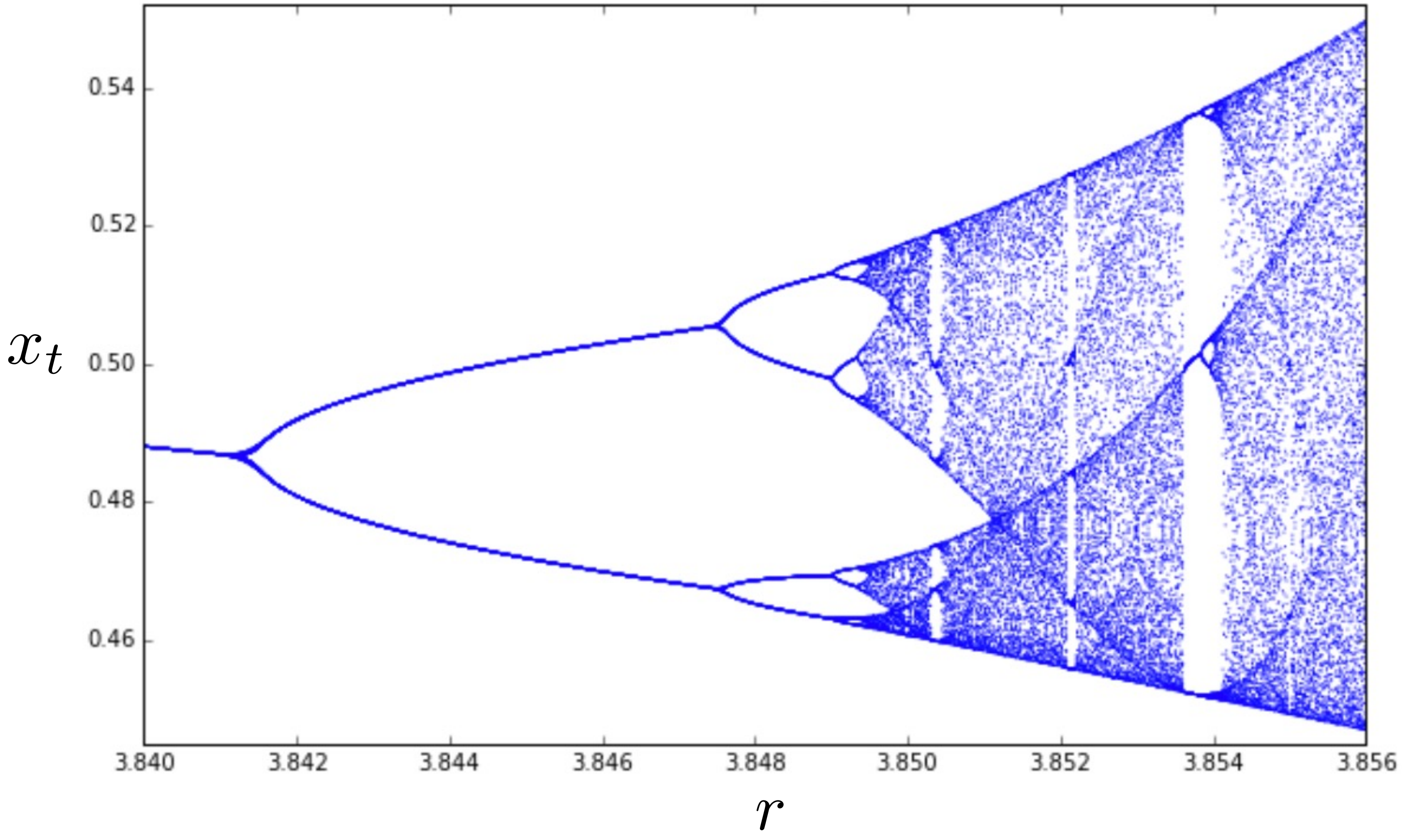
Set

$$z_t = r(1/2 - x_t)$$

$$\vdots$$

$$z_{t+1} = z_t^2 + C$$

$$C = \frac{r}{2}\left(1 - \frac{r}{2}\right)$$



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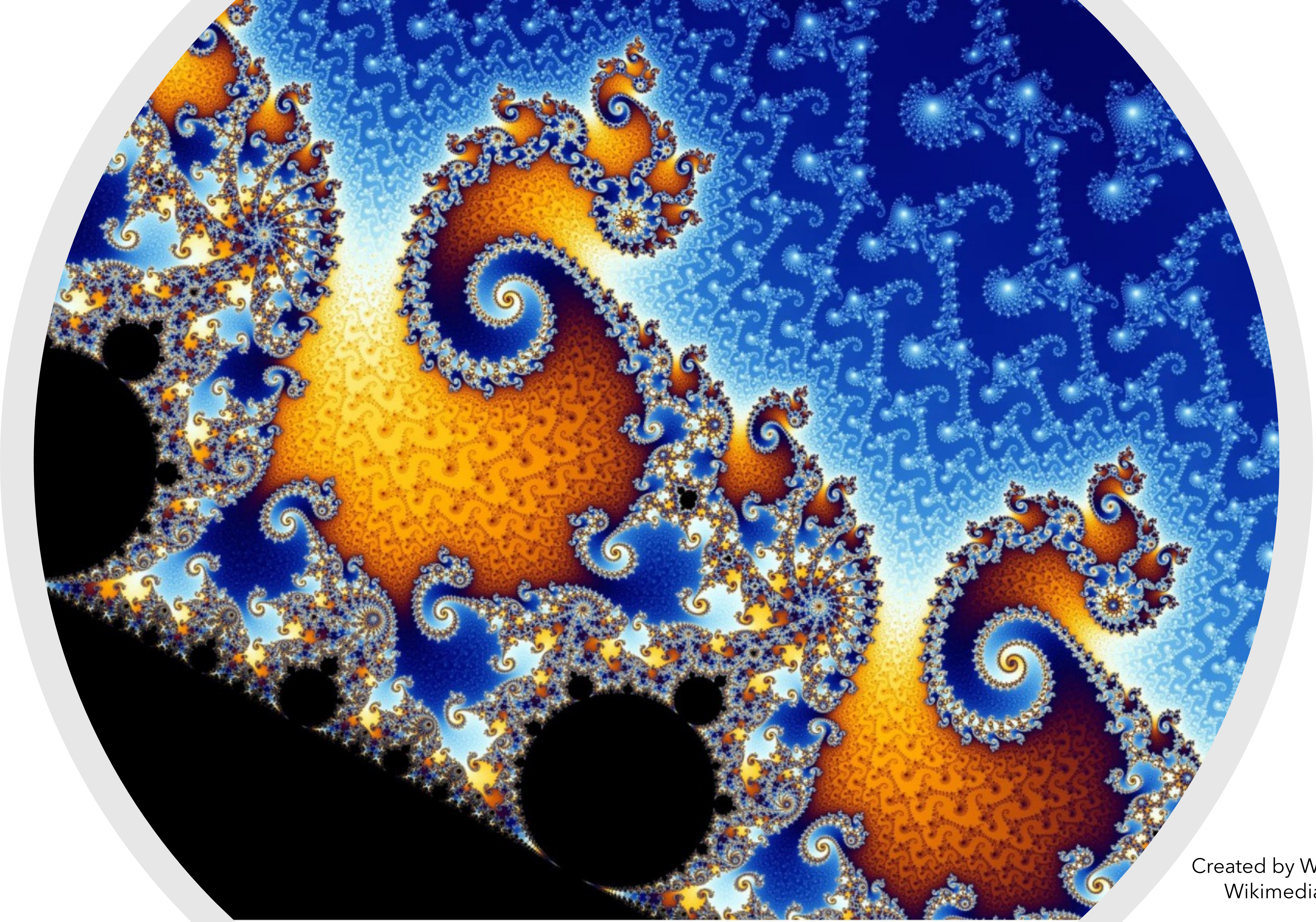


Created by Wolfgang Beyer,
Wikimedia commons

$$\frac{dx}{dt} = f(x)$$

Part II

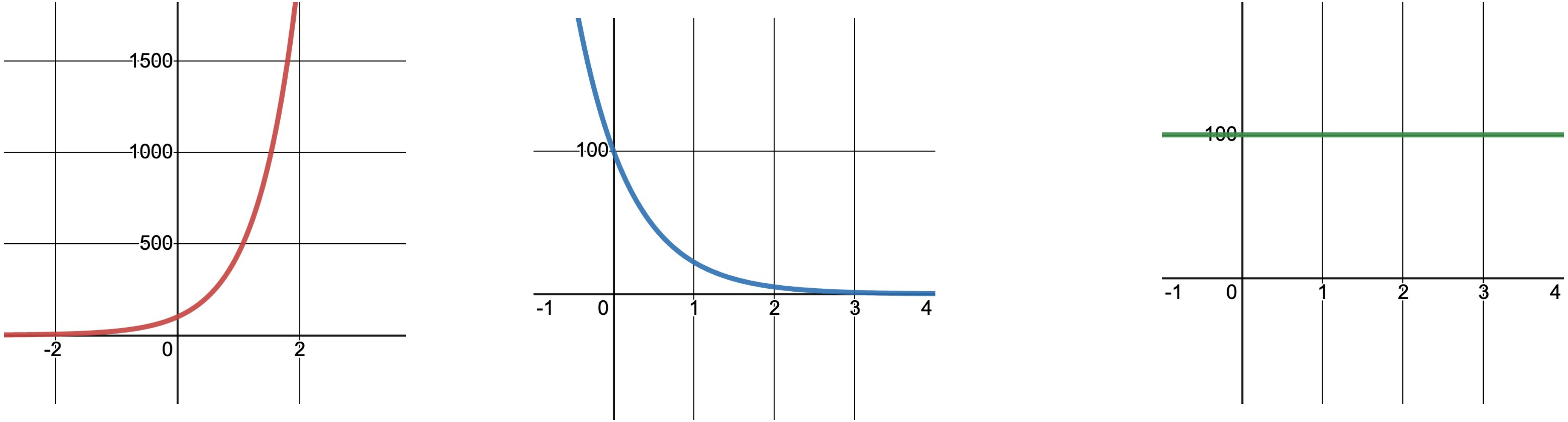
Differential equations

Linear differential equations

$$\frac{dx}{dt} = cx \qquad x(t) = x_0 e^{ct}$$

x(t) varies continuously with t

$c > 0$ Exponential growth | $c < 0$ Exponential decay | $c = 0$ Steady state / Fixed point

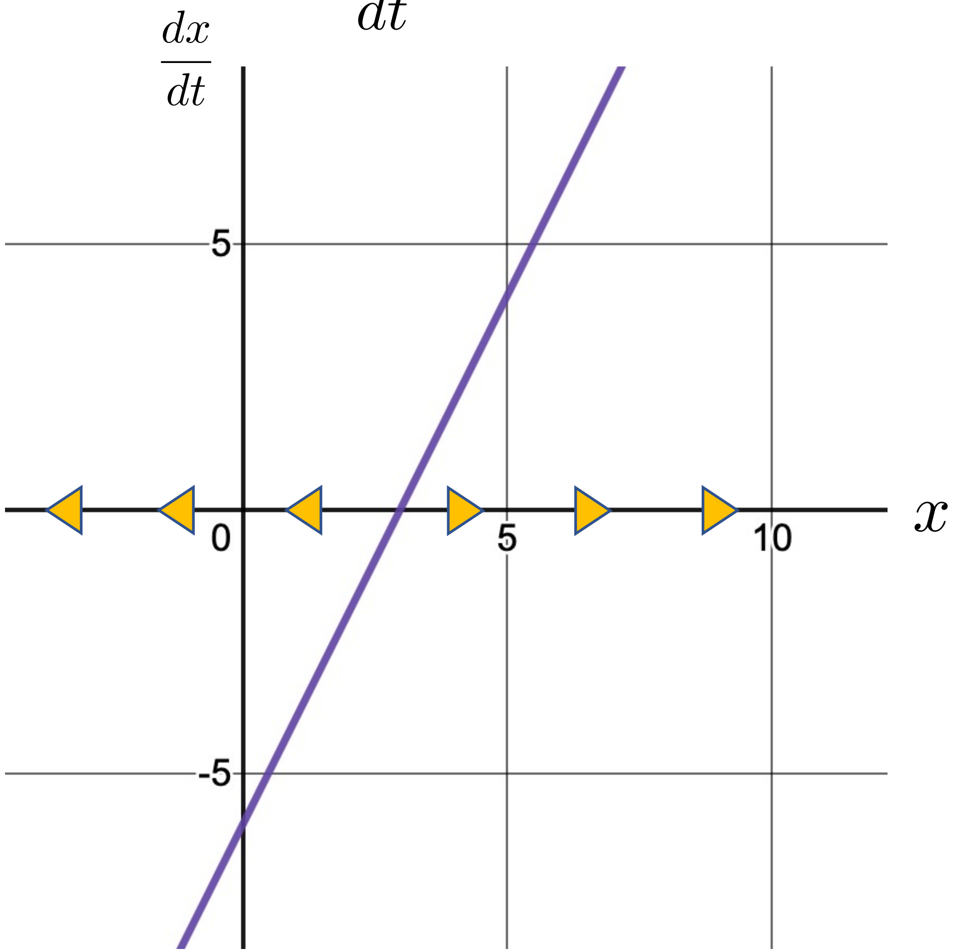


What about oscillatory dynamics? Not possible in 1D!

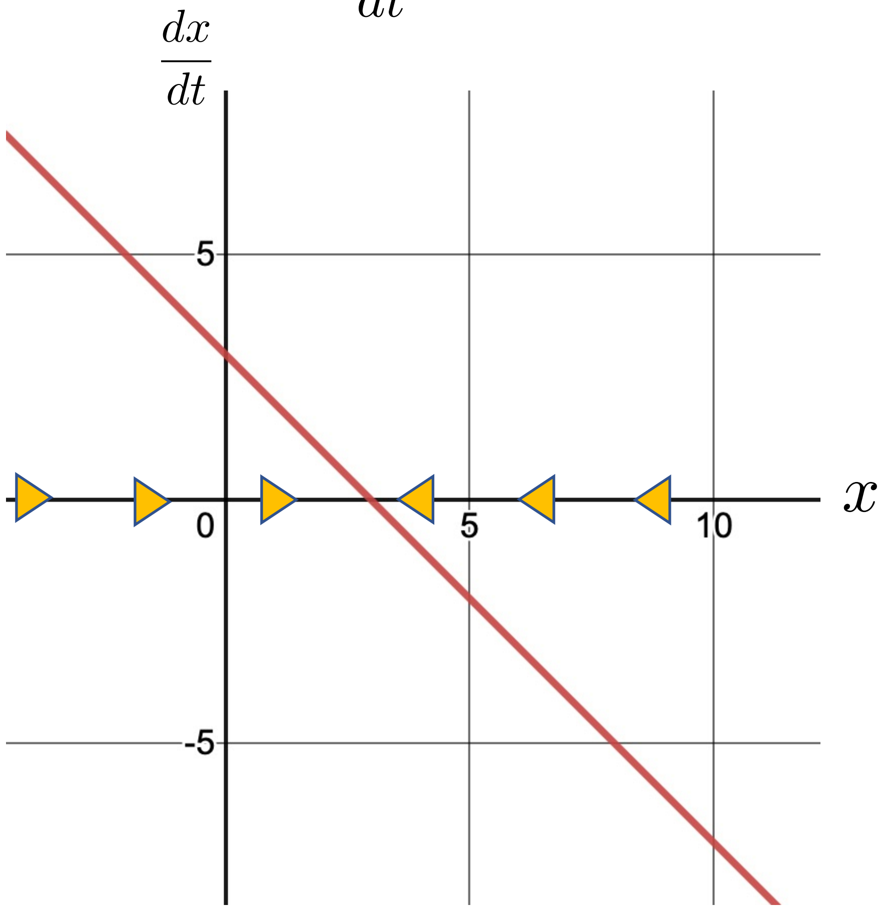
Phase plane analysis

Analytic solution can intractable – phase plane provides a quick qualitative description

$$\frac{dx}{dt} = 2x - 6$$



$$\frac{dx}{dt} = 3 - x$$



Can you guess the requirements for equilibria and stability based on this?

Equilibria and stability

x^* is an equilibrium of the differential equation $\frac{dx}{dt} = f(x)$ if and only if

$$f(x^*) = 0$$

x^* is stable if

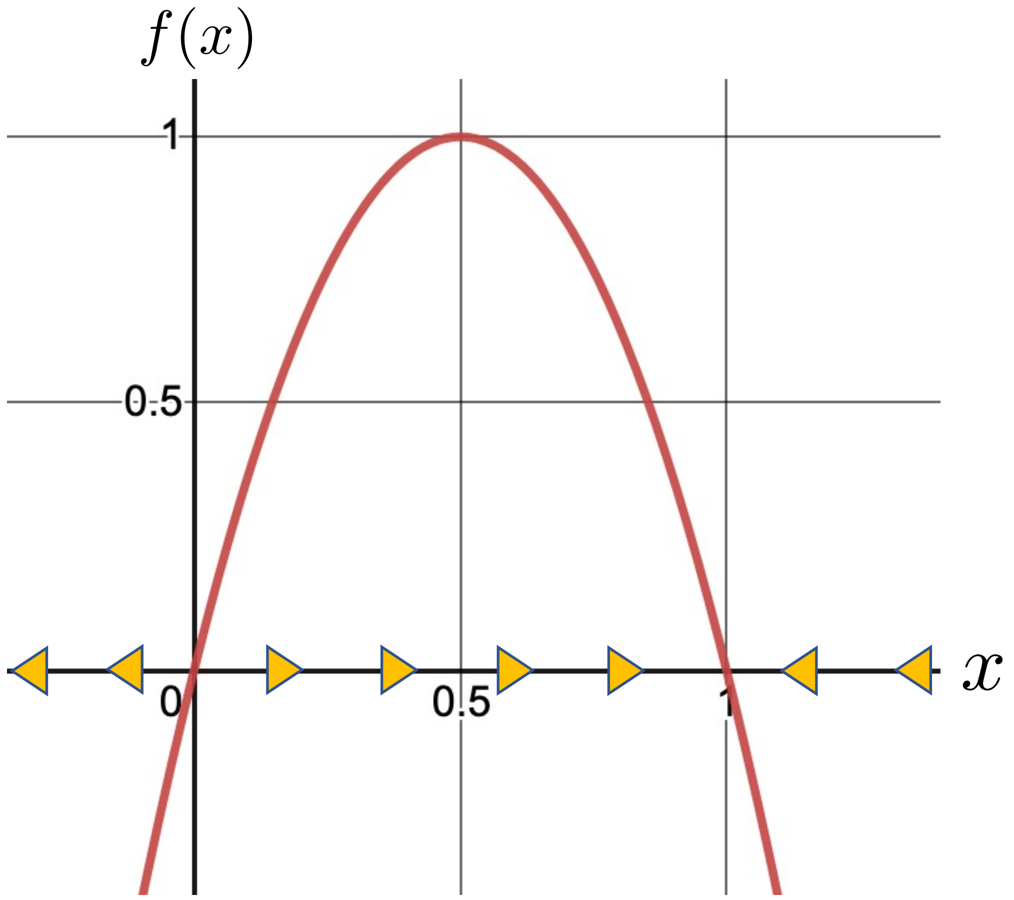
$$f'(x^*) < 0$$

x^* is unstable if

$$f'(x^*) > 0$$

E.g. continuous – time logistic equation

$$\frac{dx}{dt} = rx(1-x)$$



No oscillations (requires 2D)

No chaos (requires 3D)

Numerical solution

An approximation, unlike for discrete-time simulation which is exact.

$$\frac{dx}{dt} = f(x)$$

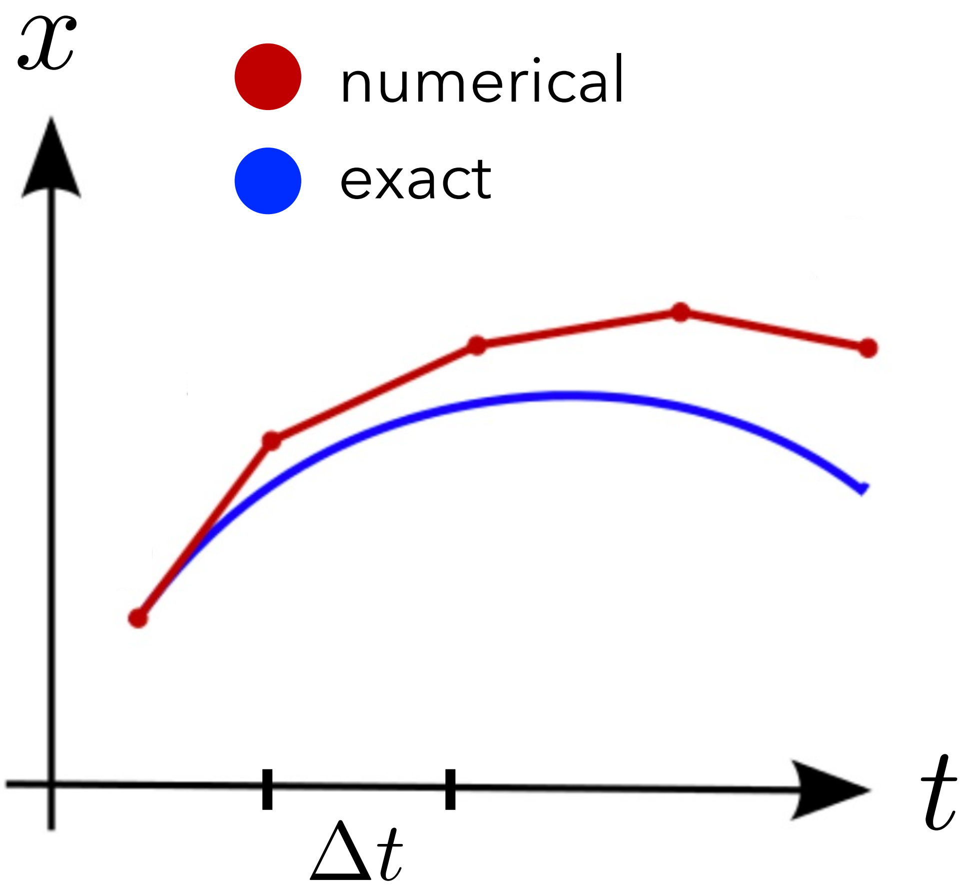
Take a small Δt

Then $$\frac{\Delta x}{\Delta t} \approx f(x)$$

$$\Delta x \approx f(x)\Delta t$$

Arrive at Euler's method

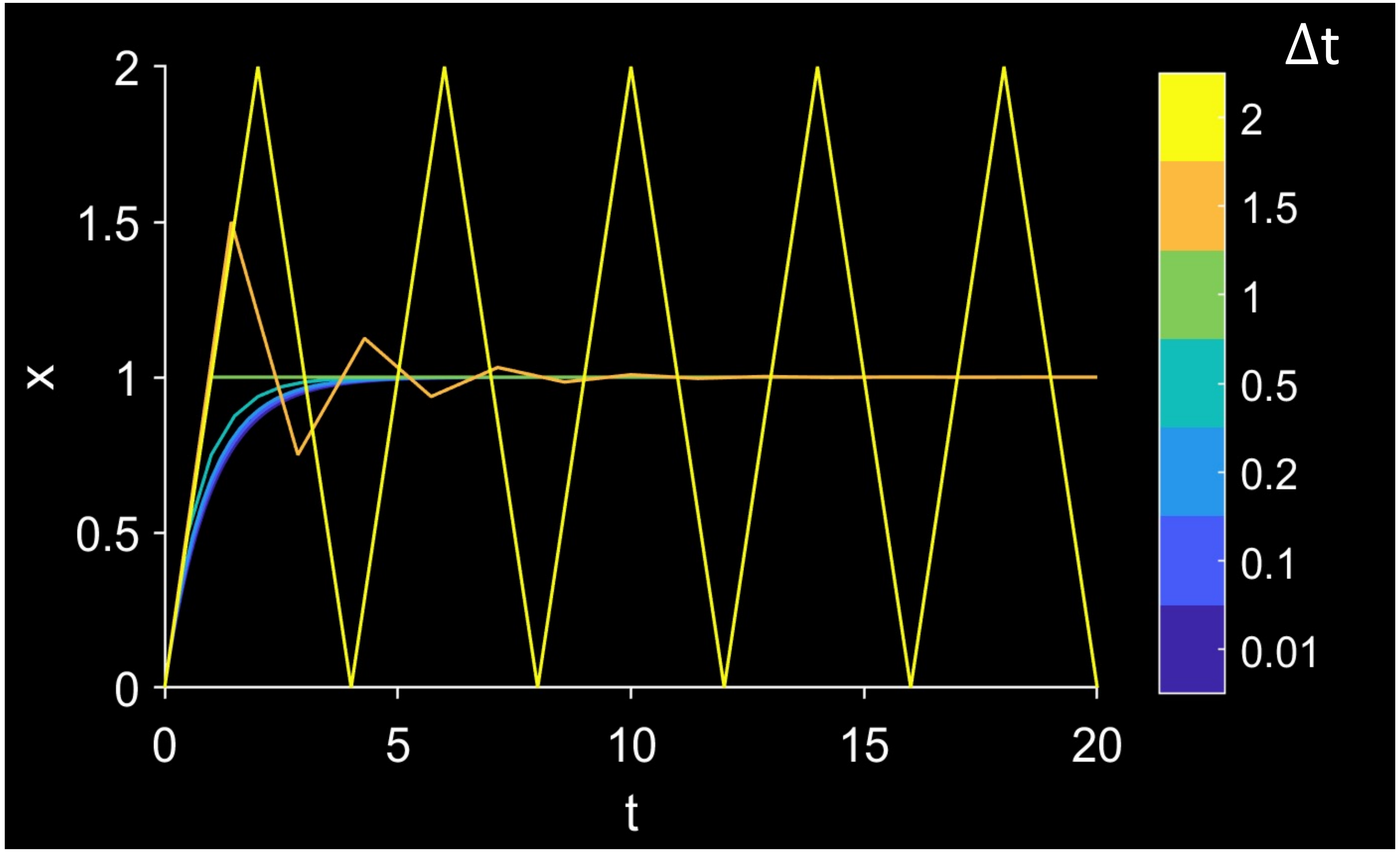
$$x_{t+1} = x_t + f(x_t)\Delta t$$



Example $\frac{dx}{dt} = 1 - x$

Choose Δt

- small enough such that dynamics stay close to true solution
- not so small that the simulation becomes too computationally costly

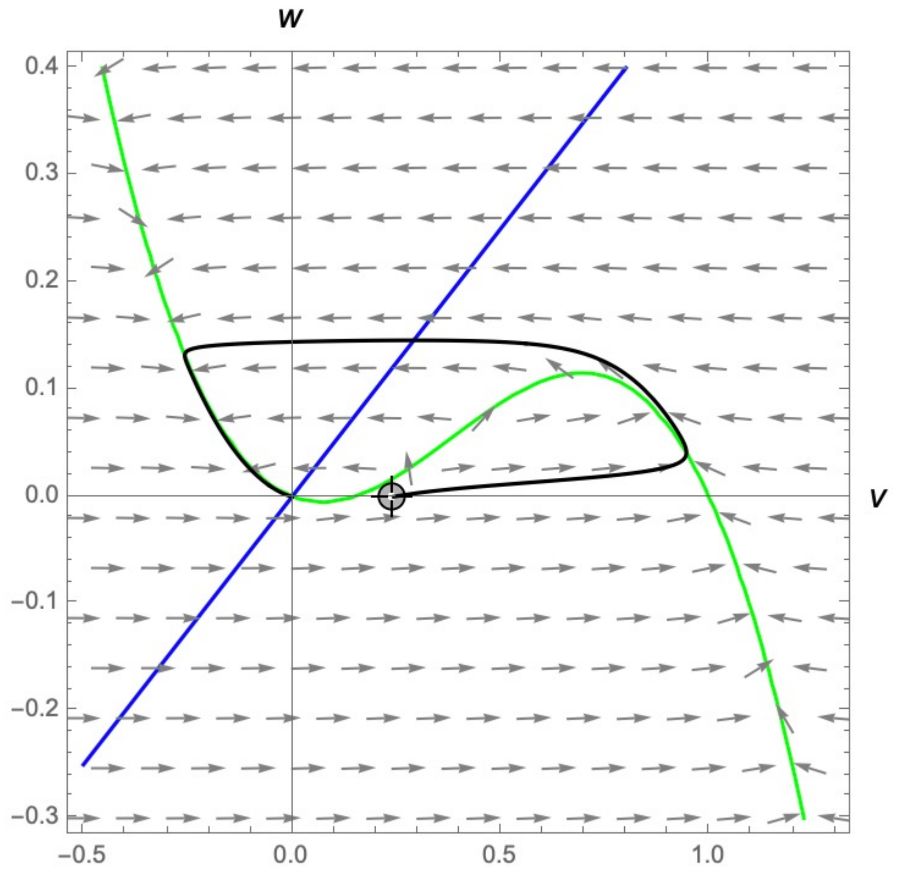


(courtesy of Niklas Brake)

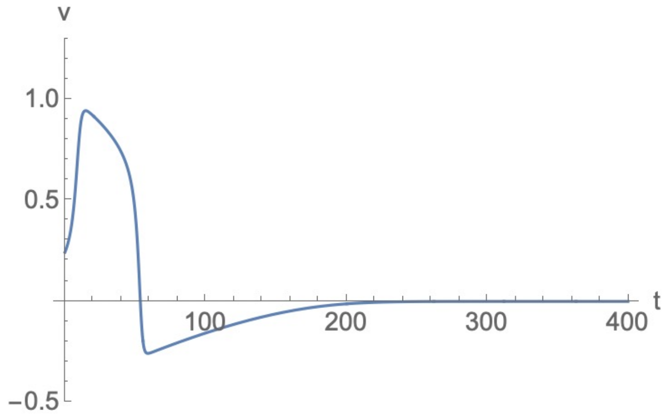
MATLAB exercise

Fitzhugh-Nagumo equations

$$\frac{dv}{dt} = -v(v-a)(v-1) - w + I_{\text{app}}$$
$$\frac{dw}{dt} = \epsilon(bv + d - cw)$$



Describe the trajectory of v based on its initial condition



Describe the qualitative changes that occur upon increasing Iapp

Physiological interpretation?